Resonances of the Laplace operator on homogeneous vector bundles on the real hyperbolic space

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Framework

$G := \mathrm{SO}_{\mathrm{e}}(n,1)$ and $\mathrm{Lie}(G) =: \mathfrak{g} = \mathfrak{so}(n,1)$
$K := \mathrm{SO}(n)$ and $\mathrm{Lie}(K) =: \mathfrak{k} = \mathfrak{so}(n)$
$\quad \hookrightarrow \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
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where $\big(g, v\big) \sim \big(gk, \tau(k^{-1})v\big)$.

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Examples:

- $E_{\mathrm{triv}_K} \simeq X \times \mathbb{C}$ → generalization of the Riemannian symmetric spaces
- $\tau_p := \Lambda^p Ad^*$ where Ad^* denote the coadjoint representation of K on $\mathfrak{p}_{\mathbb{C}}^*$, → $E_{\tau_p} = \Lambda^p H^n(\mathbb{R}) := \Lambda^p \left(T_{\mathbb{C}}^*(H^n(\mathbb{R}))\right)$.

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The smooth sections of E_τ can be identified with

$$C^\infty(G,\tau) := \left\{ f : G \xrightarrow{\text{smooth}} V_\tau \mid f(gk) = \tau(k^{-1}) f(g) \right\} \ .$$

$\mathbb{D}(E_\tau)$: set of homogeneous differential operators acting on smooth sections, i.e.

$$L(g)\ D = D\ L(g) \quad \text{for all } D \in D(E_\tau) \text{ and } g \in G$$

$$\Xi : U(\mathfrak{g}_\mathbb{C})^K \to \mathbb{D}(E_\tau)$$

$$X_1 \cdots X_n \mapsto \quad \left(f \mapsto \frac{d}{dt_1} \cdots \frac{d}{dt_n} f\left(x \exp(t_1 X_1) \cdots \exp(t_n X_n)\right)\Big|_{t_i=0} \right)$$

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Here this ring is commutative (also for $H^n(\mathbb{C})$).

Theorem (Deitmar)

Suppose G/K is connected then

$$D(E_\tau) \text{ is commutative} \quad \Leftrightarrow \quad \text{For every } \pi \in \hat{G},\ m(\tau, \pi) \leqslant 1$$

↳ Always true for $G = \mathrm{SO}_e(n,1)$ and $\mathrm{SU}(n,1)$.

(Koornwinder 1982)

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(positive) Laplacian $\Delta = \Xi(-\Omega)$ where Ω is the Casimir operator

$$\Omega := -\sum X_i^2 + \sum Y_i^2 \ ,$$

where $\{X_i\}_{i=1,...,\dim \mathfrak{k}}$ and $\{Y_i\}_{i=1,...,\dim \mathfrak{p}}$ are othonormal basis of $\mathfrak{k}$ and $\mathfrak{p}$

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It has continuous spectrum $\sigma(\Delta) = [\rho_\tau^2, +\infty[$ with $\rho_\tau^2 > 0$.

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The resolvent of Δ

$$R_\Delta(z) = (\Delta - z)^{-1}$$

is a bounded operator on $L^2(G, \tau)$ depending holomorphically on $z \in \mathbb{C} \backslash \sigma(\Delta)$, i.e.

$$\mathbb{C} \backslash \sigma(\Delta) \ni z \longrightarrow R_\Delta(z) = (\Delta - z)^{-1} \in \mathcal{B}(L^2(G, \tau)) .$$

is a holomorphic operator-valued function.

Resonances of the Laplacian: Previous works

The scalar case, i.e. when $\tau = \mathrm{triv}_K$, in rank one

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- Real hyperbolic n-space (1995): **Guillopé** and **Zworski**

Theorem
For $X = H^n(\mathbb{R})$ *and* $\Omega = \mathbb{C}$, *the resolvent* R_Δ *has:*
◇ *holomorphic extension, if* n *is odd*
◇ *meromorphic extension (with infinitely many poles) if* n *even.*

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- Meromorphic continuation on Riemannian symmetric spaces (2005): **Strohmaier**, **Mazzeo** and **Vasy**

Theorem

X = arbitrary Riemannian symmetric space of the noncompact type.
There are $\Omega \subsetneq \mathbb{C}$ open with $\sigma(\Delta) \subset \Omega$ and M Riemann surface above Ω such that

$$R_\Delta : \Omega \backslash \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$$

admits holomorphic extension to M.

Problem: Ω is not large enough to find resonances.

- Computation of resonances and residue representations for hyperbolic spaces: **Miatello** and **Will** (2000) and with a different method **Hilgert** and **Pasquale** (2009)

Theorem

X = Riemannian symmetric space of (real) rank one. Then $\zeta \mapsto R_\Delta(\zeta^2)$ extend to

- *a holomorphic function on $\mathbb{C}$ $\rightsquigarrow$ no resonances, if $X = H^n(\mathbb{R})$ with n odd,*
- *a meromorphic continuation with infinitely many (simple) poles, otherwise.*

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Resonances in higher rank

- Hilgert, Pasquale, Przedinda for $G = \mathrm{SL}(3,\mathbb{R})$, BC_2, C_2 and when X is the product of 2 rank one Riemannian symmetric spaces of non-compact type.

Harmonic analysis on vector bundles

$M := Z_K(A)$, $\mathrm{Lie}(M) = \mathfrak{m}$

For a fixed representation $\tau \in \hat{K}$:

$$\hat{M}(\tau) := \{\sigma \in \hat{M} \mid \tau|_M \supset \sigma\}$$

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For $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we denote by

$$\pi^\sigma_\lambda = \mathrm{Ind}\,^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes 1)$$

Harish-Chandra's notation for this induced representation is $\pi^\sigma_{i\lambda}$

the representation of the principal series, acting on the L^2 space $\mathscr{H}^\sigma_\lambda$.

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Frobenius reciprocity theorem: $m(\sigma, \tau|_M) = m(\tau, \pi^\sigma_\lambda) \;\; =: m_{\sigma,\tau}$

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Generalized spherical function $\varphi_\tau^{\sigma,\lambda} : G \to \mathrm{End}(V_\tau)$ defined by

$$\varphi_\tau^{\sigma,\lambda}(x) = \sum_{i=1}^{m_{\sigma,\tau}} P_i \, \pi_\lambda^\sigma(x) \, P_i^*$$

where P_i is the projection on the i-th component of τ in $\mathscr{H}_\lambda^\sigma$.

Theorem (Inversion formula, Camporesi (1998), restricted to rank one case, absolute continuous part)

For $f \in C_c^\infty(G,\tau)$*,*

$$f(x) = \sum_{\sigma \in \hat{M}(\tau)} \int_{\mathfrak{a}^*} \varphi_\tau^{\sigma,\lambda} * f(x)\ p_\sigma(\lambda)\ d\lambda$$

where $p_\sigma(\lambda)$ *is the Plancherel density (not explicitly known in arbitrary rank).*

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- Important properties of this convolution: $\varphi_\tau^{\sigma,\lambda} * f$ even (Lemma I.9.1) and rapidly decreasing in $\lambda \in \mathbb{R}$ (Corollary I.9.1)

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- $\Delta \varphi_\tau^{\sigma,\lambda} = \left(\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle - \langle \mu_\sigma + \rho_{\mathfrak{m}}, \mu_\sigma + \rho_{\mathfrak{m}} \rangle\right) \varphi_\tau^{\sigma,\lambda}$
  where $f \in C_c^\infty(G,\tau)$ and μ_σ is the highest weight of σ (Lemma I.9.2).

We define $\rho_\sigma^2 := \langle \rho, \rho \rangle - \langle \mu_\sigma + \rho_{\mathfrak{m}}, \mu_\sigma + \rho_{\mathfrak{m}} \rangle$.

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Remark: The discrete part is the part which come from discrete series representations.

$$\sum_{\gamma \in D_G} C_\gamma \int_K F^{-i\mu-\rho}(x^{-1}k)\ P_{\gamma'}\ \widetilde{f}(i\mu, k)\ dk$$

$$= \sum_{\gamma \in D_G} C_\gamma \varphi_\tau^{\gamma', -\mu} * f(x)$$

The set D_G is the set of discrete series of G. It consists of all irreducible unitary representations of G whose matrix coefficients are in $L^2(G)$.

Plancherel density

For rank one groups, it is given by Miatello (1979).
Roots system: $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n+1,\mathbb{C})$, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$, $\mathfrak{m}_{\mathbb{C}} = \mathfrak{so}(2n-1,\mathbb{C})$

$$\mathfrak{h}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}}$$

Define ϵ_j as the usual fundamental weights. The real root is ϵ_1.

$$\begin{aligned} S^0 &= \left\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in \{1,\dots,n-1\}\right\} \cup \left\{\alpha_n = \epsilon_n\right\} \\ S^0_{\mathfrak{k}_{\mathbb{C}}} &= \left\{\epsilon_i - \epsilon_{i+1} \mid i \in \{2,\dots,n\}\right\} \\ S^0_{\mathfrak{m}_{\mathbb{C}}} &= \left\{\epsilon_i - \epsilon_{i+1} \mid i \in \{2,\dots,n-1\}\right\} \cup \left\{\epsilon_n\right\} \end{aligned}$$

A fixed $(\tau, V_\tau) \in \hat{K}$ has highest weight of the form:

$$\mu_\tau = \sum_{j=2}^{n+1} a_j \epsilon_j$$

where $a_2 \geqslant \ldots \geqslant a_n \geqslant \mid a_{n+1} \mid \geqslant 0$, $a_i - a_j \in \mathbb{Z}$ and $2a_j \in \mathbb{Z}$ for all $i, j = 2, \ldots, n+1$.

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$$\mu_\sigma = \sum_{i=2}^{n} b_j \epsilon_j$$

where for all $i, j = 2, \ldots, n$ we have $a_j - b_i \in \mathbb{Z}$ and $a_2 \geqslant b_2 \geqslant a_3 \geqslant \ldots \geqslant a_n \geqslant b_n \geqslant \mid a_n \mid \geqslant 0$.

Example:

Let τ_p be the representation of the p-forms ($p \neq 0$):

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where for all $i, j = 2, \dots, n$ we have $b_i \in \mathbb{Z}$ and $1 \geqslant b_2 \geqslant 1 \geqslant \dots \geqslant 1 \geqslant b_{p+1} \geqslant 0 \geqslant b_{p+2} \geqslant \dots \geqslant 0$.

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$$\mu_\sigma = \sum_{i=2}^{p} \epsilon_j + \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\} \epsilon_{p+1}$$

We name these representations σ_{p-1} and σ_p.
One can show that $\tau_n = \sigma_{n-1} \oplus \sigma_{n-1}$

When we have the M-types in $\hat{M}(\tau)$, we can compute the Plancherel density. For σ with highest weight

$$\mu_\sigma = \sum_{i=2}^{n} b_j \epsilon_j \ ,$$

Miatello gives the Plancherel density:

$$p_\sigma(\lambda\alpha) = \left\{ \begin{array}{c} \tanh(\pi\lambda) \text{ , if } b_j \in \mathbb{Z} \\ \coth(\pi\lambda) \text{ , if } b_j \in \frac{1}{2} + \mathbb{Z} \end{array} \right\} \lambda \prod_{j=2}^{n} \left(\lambda^2 + (n + \frac{1}{2} + b_j - j)^2 \right)$$

<u>Remark</u>: $p_\sigma(\lambda\alpha)$ is even in $\lambda \in \mathbb{R}$.

Example:

For σ_l, with $l = p$ or $p - 1$,

$$\mu_{\sigma_l} = \sum_{i=2}^{l+1} \epsilon_j$$

The Plancherel density is then

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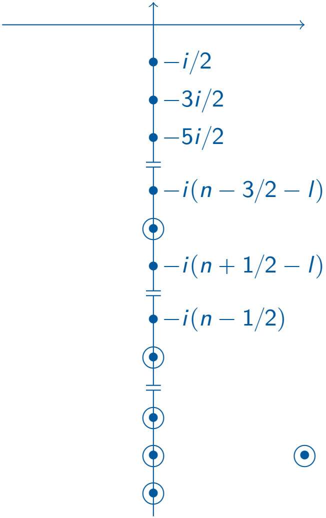

The singularities of the Plancherel density corresponding to σ_l are located at

$$\{\pm i(n-1/2-l),\ \pm i(\rho+k) \ \text{ for } k \in \mathbb{Z}_+^\times\}$$

For $k \in \mathbb{Z}_+^\times$, set

$$\lambda_k^l = -i(\rho+k) \quad \text{and} \quad \lambda_0^l = -i(n-1/2-l).$$

◉ poles

Computation of the resonances

Recall: resonances = poles of the resolvent $R(z) = (\Delta - z)^{-1}$

$$R(z)f(x) = \sum_{\sigma \in \hat{M}(\tau)} \int_{\mathfrak{a}*} \Big(\langle \lambda, \lambda \rangle + \rho_\sigma^2 - z \Big)^{-1} \varphi_\tau^{\sigma,\lambda} * f(x) \; p_\sigma(\lambda) \; d\lambda$$

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For each $\sigma \in \hat{M}(\tau)$, one gets a bunch of resonances.

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For each $\sigma \in \hat{M}(\tau)$, one gets a bunch of resonances.
We now let $\zeta_\sigma := \sqrt{-z + \rho_\sigma^2}$ and $\lambda = t\alpha$, with $t \in \mathbb{R}$.

$$\begin{aligned} \left(\langle \lambda, \lambda \rangle + \rho_\sigma^2 - z\right)^{-1} &= \left(t^2|\alpha|^2 - \zeta_\sigma^2\right)^{-1} \\ &= \frac{1}{2t|\alpha|} \left((t|\alpha| - \zeta_\sigma)^{-1} + (t|\alpha| + \zeta_\sigma)^{-1}\right) \end{aligned}$$

Computation of the resonances

Recall: resonances = poles of the resolvent $R(z) = (\Delta - z)^{-1}$

$$R(z)f(x) = \sum_{\sigma \in \hat{M}(\tau)} \underbrace{\int_{\mathfrak{a}^*} \Big(\langle \lambda, \lambda \rangle + \rho_\sigma^2 - z \Big)^{-1} \varphi_\tau^{\sigma,\lambda} * f(x) \; p_\sigma(\lambda) \; d\lambda}_{R_\sigma(z)f(x)}$$

For each $\sigma \in \hat{M}(\tau)$, one gets a bunch of resonances.
We now let $\zeta_\sigma := \sqrt{-z + \rho_\sigma^2}$ and $\lambda = t\alpha$, with $t \in \mathbb{R}$.

$$\begin{aligned} \Big(\langle \lambda, \lambda \rangle + \rho_\sigma^2 - z \Big)^{-1} &= \Big(t^2 |\alpha|^2 - \zeta_\sigma^2 \Big)^{-1} \\ &= \frac{1}{2t|\alpha|} \Big((t|\alpha| - \zeta_\sigma)^{-1} + (t|\alpha| + \zeta_\sigma)^{-1} \Big) \end{aligned}$$

$$R_\sigma(\zeta_\sigma)f(x) = \frac{1}{|\alpha|} \int_{\mathbb{R}} \frac{1}{\lambda|\alpha| - \zeta_\sigma} \Big(\varphi_\tau^{\sigma,\lambda\alpha} * f \Big)(x) \; \frac{p_\sigma(\lambda\alpha)}{\lambda} \; d\lambda$$

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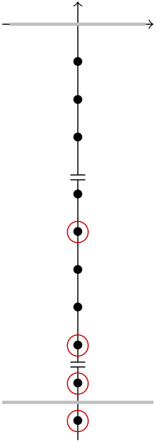
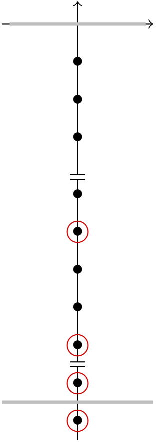


$-i(N + \frac{1}{4})$



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&= \frac{1}{|\alpha|}\int_{\mathbb{R}-i(N+1/4)} \frac{1}{\zeta_\sigma-\lambda|\alpha|}\left(\varphi_\tau^{\sigma,\lambda\alpha} * f\right)(x)\ \frac{p_\sigma(\lambda\alpha)}{\lambda}\ d\lambda \\
&- \frac{2i\pi}{|\alpha|}\sum_{\substack{k\in\mathbb{N}_\sigma \\ \lambda_k>-i(N+1/4)}} \frac{1}{\zeta_\sigma-\lambda_k|\alpha|}\left(\varphi_\tau^{\sigma,\lambda_k\alpha} * f\right)(x)\ \operatorname*{Res}_{\lambda=\lambda_k} \frac{p_\sigma(\lambda\alpha)}{\lambda}
\end{aligned}
$$

$-i(N+\frac{1}{4})$

Theorem

The resonances of the Laplace operator acting on the sections of E_τ appear in families parametrized by the elements of $\hat{M}(\tau)$. Let

$$S_\sigma = \left\{ (z,\zeta) \in \mathbb{C}^2 \mid \zeta^2 := -z - \langle \rho, \rho \rangle + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M \rangle) \right\} \ .$$

Then the resolvent extends meromorphically from $S_\sigma^+ = \{(z,\zeta) \in S \mid \Im(\zeta) > 0\}$ to S_σ. The (simple) poles of this extension are the pairs

$$(z_{\sigma,k}, \zeta_{\sigma,k}) = \left((B_{\max}+k)^2 |\alpha|^2 - \rho_\alpha^2 |\alpha|^2 + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M \rangle \, , -i(B_{\max}+k)^2 |\alpha|^2 \ \right)$$

where $k \in \mathbb{N}_\sigma$,μ_σ is the highest weight of the representation σ, $B_{\max}$ depends on μ_σ and ρ_M is the half sum of positive roots for $\mathfrak{m}$.

Example: We obtain a meromorphic extension of the resolvant on the half plane $\Im(\zeta_l) > -(\rho + N)$, for every $N \in \mathbb{Z}_+$. We have up to a constant

$$\left(R_l(\zeta_l)f\right)(x) = \left(R_{l,N}(\zeta_l)f\right)(x) + \frac{2i\pi}{|\alpha|}\left(\sum_{k=0}^{N}\left(\lambda_k^l|\alpha| - \zeta_l\right)^{-1}\left(\varphi_\tau^{\sigma_l,\lambda_k^l\alpha} * f\right)(x)\right)$$

where

$$\left(R_{l,N}(\zeta_l)f\right)(x) =$$

$$\int_{\mathbb{R}-i\left(\rho+N+\frac{1}{2}\right)}\left(t|\alpha| - \zeta_l\right)^{-1}\left(\varphi_\tau^{\sigma_l,t\alpha} * f\right)(x)\tanh(\pi t)\prod_{k=1}^{n}\frac{\left(t^2 + \left(k - \frac{1}{2}\right)^2\right)}{t^2 + (\rho - p)^2}dt$$

Residue representations

For each point λ_k:

$$\mathscr{E}_k^{\sigma} := \{\varphi_\tau^{\sigma,\lambda_k} * f \mid f \in C_c^\infty(G,\tau)\} \curvearrowleft^{L} G$$

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$$\begin{array}{rccc} R_k^\sigma : & C_c^\infty(G,\tau) & \longrightarrow & C^\infty(G,\tau) \\ & f & \longmapsto & \varphi_\tau^{\sigma,\lambda_k} * f \end{array}$$

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↳ What representation is it?

$$\begin{array}{cccc} R_k^\sigma : C_c^\infty(G,\tau) & \to & \mathscr{H}^\sigma_{\lambda_k \alpha} & \to C^\infty(G,\tau) \\ f & \mapsto & T_l(f) & \mapsto \sum\limits_l P_l \, \pi^\sigma_{\lambda_k \alpha}(\,\cdot\,^{-1})\big(T_l(f)\big) \end{array}$$

↳ Intertwining maps ↵

where for each $l = 1, \ldots, m(\sigma,\tau)$, the map $T_l : C_c^\infty(G,\tau) \to \mathscr{H}^\sigma_{\lambda_k \alpha}$ is defined by

$$T_l(f) = \int_G \pi^\sigma_{\lambda_k}(g)\big(P_l^* f(g)\big) \, dg \; .$$

Lemma

T_l is an intertwining operator between the left regular representation on $C_c^\infty(G,\tau)$ and the principal series representation $(\pi^\sigma_{\lambda_k}, \mathscr{H}^\sigma_{\lambda_k\alpha})$. Moreover, for each l the range of the map T_l is the closed subspace of $\mathscr{H}^\sigma_{\lambda_k\alpha}$ spanned by the left translates of $P_l^* V_\tau$. We will denote this space by $\langle \pi^\sigma_{\lambda_k}(G) P_l^* V_\tau \rangle$.

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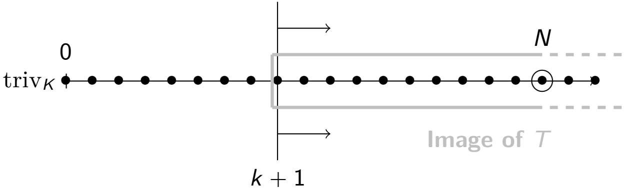
- $\rightsquigarrow$ $T_l\big(C_c^\infty(G,\tau)\big)$ is a subrepresentation of $\mathscr{H}^\sigma_{\lambda_k\alpha}$
- $\rightsquigarrow$ As Poisson transform = intertwining map
  $\Rightarrow$ if $m(\sigma,\tau)=1$, then $\mathscr{E}_k = {}^{\mathrm{Im}(T)}/_{\mathrm{Ker}(\text{Poisson tr.})}$
- $\rightsquigarrow$ Structure of $\mathscr{H}^\sigma_{\lambda_k\alpha}$ is not known in general
  $\Rightarrow$ we restrict to $\sigma = \mathrm{triv.}$ Then $\tau|_M \subset \sigma \quad \Leftrightarrow \quad \tau = \tau_n$ with highest weight $\mu_N = N\epsilon_1$

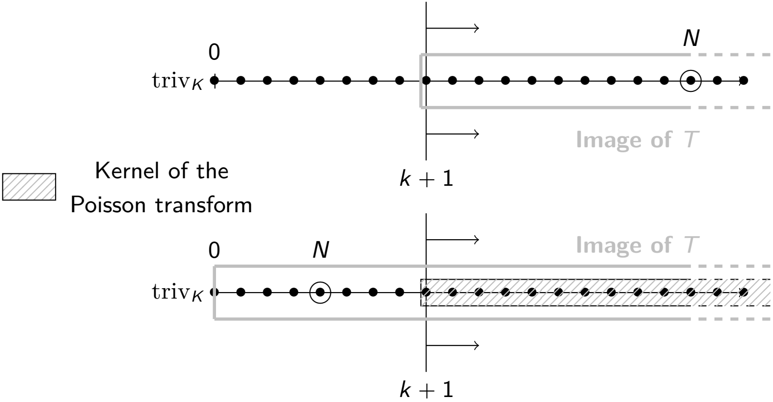
The principal series representation



Frobenius $\Rightarrow$ $\left(\tau \supset \mathrm{triv} \quad \Leftrightarrow \quad \tau \subset \mathscr{H}_{\lambda_k \alpha}\right)$

(Howe-Tan[1993])

The principal series representation

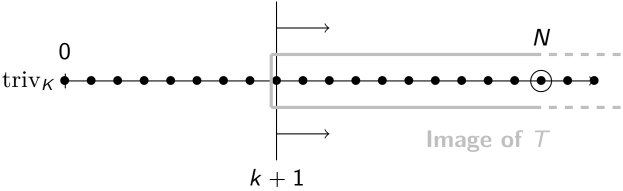


Recall : Langlands parameters are:

- a cuspidal parabolic subgroup MAN
- a discrete series representation of M
- a linear form $\nu \in \mathfrak{a}^*$ with $\Re(\nu) > 0$.

Then a representation with Langlands parameters (MA, σ, ν), is the unique irreducible quotient of $\mathscr{H}_\nu^\sigma = \mathrm{Ind}_{MAN}^G(\sigma \otimes e^{i\nu} \otimes \mathrm{triv})$.

1. Find the minimal K-type $\tau_{\min}$ of $\mathscr{E}_k$
2. Find the unique (if it exists) $\sigma \in \hat{M}$, such that $\mathrm{Ind}_M^K(\sigma)$ has $\tau_{\min}$ as minimal (one) K-type.
3. Compare the regular characters of $\mathscr{H}_\nu^\sigma$ and $\mathscr{H}_{\lambda_k}$ to find the parameter ν.



A minimal K-type minimizes the Vogan norm of the highest weight in $\mathscr{E}_k$:

$$\|\mu_I\|_V = \langle \mu_I + 2\rho_K, \mu_I + 2\rho_K \rangle$$

where ρ_K is half sum of positive roots in $S_+^{\mathfrak{k}_\mathbb{C}}$.
Here it's clear that $\tau_{\min} = \tau_{k+1}$. Then $\sigma \subset \tau_{\min}|_M$ if and only if

$$\mu_\sigma(a) = a\epsilon_1 \ ,$$

where $a \in [\![0, k+1]\!]$. The only one which has the same minimal K-type is $\mu_\sigma(k+1)$. Denote this represention σ_{k+1}

To find ν one has to compare the infinitesimal character of

$$\mathrm{Ind}_{MAN}^{G}(\mathrm{triv} \otimes e^{(\rho_\alpha + k)\alpha} \otimes \mathrm{triv}) \quad \text{and} \quad \mathrm{Ind}_{MAN}^{G}(\sigma_{k+1} \otimes e^{\nu} \otimes \mathrm{triv}) \ .$$

They have to agree up to the action of the Weyl group of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$.

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$$\mathrm{Ind}_{MAN}^{G}\big(1\otimes e^{i\lambda_k\alpha}\otimes 1\big)\to\lambda_k\epsilon_1+\rho_{\mathfrak{m}}$$
$$\mathrm{Ind}_{MAN}^{G}\big(\sigma_{k+1}\otimes e^{\nu\alpha}\otimes 1\big)\to\nu\epsilon_1+\mu_{\sigma\,k+1}+\rho_{\mathfrak{m}}$$

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$$\begin{aligned}\mathrm{Ind}_{MAN}^{G}\big(1\otimes e^{i\lambda_k\alpha}\otimes 1\big) &\to (n-1/2+k)\epsilon_1\\ &\quad+\sum_{2}^{n+1}(n-j+\frac{1}{2})\epsilon_j\\ \mathrm{Ind}_{MAN}^{G}\big(\sigma_{k+1}\otimes e^{\nu\alpha}\otimes 1\big) &\to \nu\epsilon_1+\frac{k+1}{2}(\epsilon_1-\epsilon_2)+(k+1)\epsilon_3\\ &\quad+(1+k)\epsilon_2+\sum_{2}^{n+1}(n-j+\frac{1}{2})\epsilon_j\end{aligned}$$

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$$\begin{aligned}
\mathrm{Ind}_{MAN}^{G}\big(1\otimes e^{i\lambda_k\alpha}\otimes 1\big) &\to (n-1/2+k)\epsilon_1+(n-\frac{3}{2})\epsilon_2 \\
&+\sum_{3}^{n+1}(n-j+\frac{1}{2})\epsilon_j \\
\mathrm{Ind}_{MAN}^{G}\big(\sigma_{k+1}\otimes e^{\nu\alpha}\otimes 1\big) &\to \nu\epsilon_1+(n-\frac{1}{2}+k)\epsilon_2 \\
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Weyl group action ⇔ permutations of ϵ_j's and sign changes. So

$$\to \boxed{\nu=\pm\left(n+\frac{3}{2}\right)}$$

Langlands parameters: (MA, σ_{k+1}, ν) with minimal K-type $\tau_{\min}$.

Langlands parameters of $\mathscr{E}_k$ when $G = \mathrm{SO}(2n,1)$

Theorem:

Case	Minimal K-type	σ	values of ν
$N > k+1$	τ_{k+1}	σ_{k+1}	$\pm\left(n - \frac{3}{2}\right)\alpha$
$N \leqslant k$	triv_K	triv_M	$(\rho_\alpha + k)\alpha$

Example:
Just one case is treated by this Theorem : $\tau_1\big|_M = \sigma_0 \oplus \sigma_1$
Only one infinite dimensional representation $\rightarrow \mathscr{E}_0$.

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For semisimple Lie groups, π admissible representation of G,
$\mathrm{WF}(\pi) =$ closure of a union of nilpotent orbits in $\mathfrak{g}$ (under $\mathrm{Ad}(G)$)

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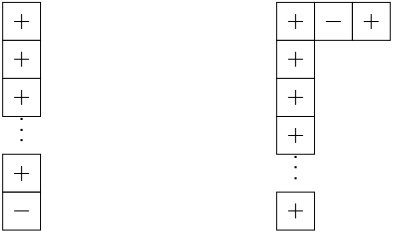
For $\mathrm{SO}(2n, 1)$, there are 2 nilpotent orbits



with $2n$ '$+$', which respectively correspond respectively to the 0 orbit and the orbit generated by $\mathfrak{g}_\alpha$.

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Young diagram with $2n + 1$ squares $\rightsquigarrow$ partition $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$ of $2n + 1$

$$\rightsquigarrow \dim = (2n+1)^2 - \frac{1}{2}\sum s_i^2 - \frac{1}{2}\sum_{\text{odd}} r_i$$

where $s_i = |\{j \mid d_j \geqslant i\}|$ and $r_i = |\{j \mid d_j = i\}|$ in a partition $[d_1, \ldots, d_k]$ of $2n + 1$.

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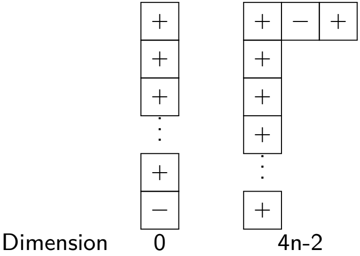

Dimension of the wave front set $= 2\times$ **Gelfand-Kirillov dimension**
$\Rightarrow$ G-K dimension is $2n-1 \Rightarrow \mathrm{WF}$ has dimension $4n-2$
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Corollary:

$\mathscr{E}_k \quad \Leftrightarrow \quad$ Components of $\mathscr{H}_{\lambda_k \alpha} \quad \Leftrightarrow \quad$ nilpotents orbits

Rank one

G	K	$X = G/K$	Σ^+
$\mathrm{Spin}(n,1)$	$\mathrm{Spin}(n)$	real h. s.	
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	complex h. s.	
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	quaternionic h. s.	
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	octonionic h. p.	