Resonances of the Laplace operator on homogeneous vector bundles on rank-one symmetric spaces

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Introduction

Schrödinger (1926): Hydrogen Stark Hamiltonian

$$\Delta + W$$

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For the Laplace operator:

$$R(z) = (\Delta - z)^{-1} \quad \rightharpoondown L^2$$

Framework

G: non-compact semisimple Lie group, finite center, $\mathrm{Lie}(G) = \mathfrak{g}$
K: maximal compact of G, $\mathrm{Lie}(K) = \mathfrak{k}$, so $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
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- $E_{\mathrm{triv}_K} \simeq X \times \mathbb{C}$ $\rightarrow$ generalization of the Riemannian symmetric spaces
- $G = \mathrm{SO_e}(1,n)$, $K = \mathrm{SO}(n)$ and $\tau_p := \Lambda^p Ad^*$ where Ad^* denote the coadjoint representation of K on $\mathfrak{p}_{\mathbb{C}}^*$,
  $\rightarrow$ $E_{\tau_p} = \Lambda^p H^n(\mathbb{R}) := \Lambda^p \left(T^*_{\mathbb{C}}(H^n(\mathbb{R}))\right)$.

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The smooth sections of E_τ can be identified with

$$C^\infty(G,\tau) := \left\{ f : G \xrightarrow{\text{smooth}} V_\tau \mid f(gk) = \tau(k^{-1}) f(g) \right\} .$$

$\mathbb{D}(E_\tau)$: set of homogeneous differential operators acting on smooth sections, i.e.

$$L(g)\ D = D\ L(g) \quad \text{for all } D \in D(E_\tau) \text{ and } g \in G$$

$$\Xi : U(\mathfrak{g}_{\mathbb{C}})^K \longrightarrow \mathbb{D}(E_\tau)$$

$$X_1 \cdots X_n \mapsto \quad \left(f \mapsto \frac{d}{dt_1} \cdots \frac{d}{dt_n} f\left(x \exp(t_1 X_1) \cdots \exp(t_n X_n)\right)\Big|_{t_i=0} \right)$$

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(positive) Laplacian $\Delta = \Xi(-\Omega)$ where Ω is the Casimir operator

$$\Omega := -\sum X_i^2 + \sum Y_i^2 \ ,$$

where $\{X_i\}_{i=1,...,\dim \mathfrak{k}}$ and $\{Y_i\}_{i=1,...,\dim \mathfrak{p}}$ are othonormal basis of $\mathfrak{k}$ and $\mathfrak{p}$

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The resolvent of Δ

$$R_\Delta(z) = (\Delta - z)^{-1}$$

is a bounded operator on $L^2(G,\tau)$ depending holomorphically on $z \in \mathbb{C} \backslash \sigma(\Delta)$, i.e.

$$\mathbb{C} \backslash \sigma(\Delta) \ni z \longrightarrow R_\Delta(z) = (\Delta - z)^{-1} \in \mathcal{B}(L^2(G,\tau)) \, .$$

is a holomorphic operator-valued function.

Harmonic analysis on vector bundles

$M := Z_K(A)$, $\mathrm{Lie}(M) = \mathfrak{m}$

For a fixed representation $\tau \in \hat{K}$:

$$\hat{M}(\tau) := \{\sigma \in \hat{M} \mid \tau|_M \supset \sigma\}$$

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For $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we denote by

$$\pi^\sigma_\lambda = \mathrm{Ind}\,^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes 1)$$

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Generalized spherical function $\varphi^{\sigma,\lambda}_\tau : G \to \text{End}(V_\tau)$ defined by

$$\varphi^{\sigma,\lambda}_\tau(x) = \sum_{i=1}^{m_{\sigma,\tau}} P_i\ \pi^\sigma_\lambda(x)\ P_i^*$$

where P_i is the projection on the i-th component of τ in $\mathscr{H}^\sigma_\lambda$.

Rank one

Assumption in this thesis: G is of real rank one $\iff$ $\dim \mathfrak{a} = 1$

G	K	$X = G/K$
$\mathrm{Spin}(n,1)$	$\mathrm{Spin}(n)$	real h. s.
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$	complex h. s.
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	quaternionic h. s.
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	octonionic h. p.

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- Real hyperbolic n-space (1995): **Guillopé** and **Zworski**

Theorem

For $X = H^n(\mathbb{R})$ *and* $\Omega = \mathbb{C}$*, the resolvent* R_Δ *has:*

◇ *holomorphic extension, if n is odd*
◇ *meromorphic extension (with infinitely many poles) if n even.*

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- Meromorphic continuation on Riemannian symmetric spaces (2005): **Strohmaier**, **Mazzeo** and **Vasy**

Theorem

X = arbitrary Riemannian symmetric space of the noncompact type.
There are $\Omega \subsetneq \mathbb{C}$ open with $\sigma(\Delta) \subset \Omega$ and M Riemann surface above Ω such that

$$R_\Delta : \Omega \backslash \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$$

admits holomorphic extension to M.

Problem: Ω is not large enough to find resonances.

- Computation of resonances and residue representations for hyperbolic spaces: **Miatello** and **Will** (2000) and with a different method **Hilgert** and **Pasquale** (2009)

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X = Riemannian symmetric space of (real) rank one. Then $\zeta \mapsto R_\Delta(\zeta^2)$ extend to

- *a holomorphic function on $\mathbb{C}$ $\rightsquigarrow$ no resonances, if $X = H^n(\mathbb{R})$ with n odd,*
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Resonances in higher rank

- Hilgert, Pasquale, Przedinda for $G = \mathrm{SL}(3, \mathbb{R})$, BC_2, C_2 and when X is the product of 2 rank one Riemannian symmetric spaces of non-compact type.

First result gives explicitely the meromorphic continuation of the resolvant and the resonances for τ arbitrary. They depend on the M-types σ which occurs in $\tau|_M$. Let $\hat{M}(\tau)$ be this set.

Theorem

The resonances of the Laplace operator acting on the sections of E_τ appear in families parametrized by the elements of $\hat{M}(\tau)$. Let

$$S_\sigma = \left\{(z,\zeta) \in \mathbb{C}^2 \mid \zeta^2 := -z - \langle \rho, \rho \rangle + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M \rangle)\right\} .$$

Then the resolvent extends meromorphically from $S_\sigma^+ = \{(z,\zeta) \in S \mid \Im(\zeta) > 0\}$ to S_σ. The (simple) poles of this extension are the pairs

$$(z_{\sigma,k}, \zeta_{\sigma,k}) = ((B_{\max} + k)^2 |\alpha|^2 - \rho_\alpha^2 |\alpha|^2 + \langle \mu_\sigma + \rho_M, \mu_\sigma + \rho_M \rangle \,, -i(B_{\max} + k)^2 |\alpha|^2 \,)$$

where $k \in \mathbb{N}_\sigma$,μ_σ is the highest weight of the representation σ, $B_{\max}$ depends on μ_σ and ρ_M is the half sum of positive roots for $\mathfrak{m}$.

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Second question:

Residue at the resonances : $\mathrm{E}_k^\sigma := \{\varphi_\tau^{\sigma,\lambda_k} * f \mid f \in C_c^\infty(G,\tau)\} \curvearrowleft^L G$

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What is the representation obtained?

To have a more general result, we suppose that τ contain the trivial representation of M.

Theorem

The residue representations $\mathscr{E}_k$ are ***all irreducible****. We found which one are* ***unitary****. We found their* ***Langlands parameters*** *: these give a full description of the $(\mathfrak{g}, K)$-module as unique irreducible quotient of a principal series representation. There are* ***discrete series representations****.*

Wave front set

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$\mathrm{WF}(\pi)$ = closure of a union of nilpotent orbits in $\mathfrak{g}$ (under $\mathrm{Ad}(G)$)

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Corollary:

$\mathscr{E}_k \quad \Leftrightarrow \quad$ Components of $\mathscr{H}_{\lambda_k \alpha} \quad \Leftrightarrow \quad$ nilpotents orbits