Resonances of the d'Alembertian on the Anti-de Sitter space $\mathrm{SO_e}(2,2)/\mathrm{SO_e}(2,1)$

ROBY Simon
Ph.D from Université de Lorraine, Metz, France (2021)
Advisor: 刘正伟 (LIU Zhengwei)

October 15th 2023

丘成桐数学科学中心
YAU MATHEMATICAL SCIENCES CENTER

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(Geometric)

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Riemannian $\rightarrow H^n(\mathbb{R}) = \mathrm{SO_e}(n,1)/\mathrm{SO}(n)$, $\quad X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$

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Two worlds to study "*resonances*":

- "Quantum" world:
  Poles of meromorphic continuation of operators acting on functions
- "Classical" world:
  Properties of the flow of Hamiltonian vector fields X_f, with f being a function on T^*X

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Spectral properties of operator $\xleftrightarrow{\text{Policott-Ruelle resonances}}$ geodesic flow

$\mathfrak{g}$: Lie Algebra of G ($\mathfrak{g} = T_e(G)$)
Cartan decomposition: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$
$\mathfrak{g}_{\mathbb{C}}$: Complexification
$U(\mathfrak{g}_{\mathbb{C}})$: Universal enveloping algebra
$\mathbb{D}(X)$: space of differential operators acting on functions on X,

$$\Xi : U(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{D}(X)$$

$$X_1 \cdots X_n \mapsto \left(f \mapsto \frac{d}{dt_1} \cdots \frac{d}{dt_n} f\left(x \exp(t_1 X_1) \cdots \exp(t_n X_n)\right)\Big|_{t_i=0} \right)$$

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In $U(\mathfrak{g}_{\mathbb{C}})$, define Ω the Casimir operator

$$\Omega := -\sum X_i^2 + \sum Y_i^2 \ ,$$

where $\{X_i\}_{i=1,...,\dim \mathfrak{k}}$ and $\{Y_i\}_{i=1,...,\dim \mathfrak{p}}$ are othonormal basis of $\mathfrak{k}$ and $\mathfrak{p}$

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For Riemannian symmetric spaces $\longrightarrow$ Laplacian $\Delta = \Xi(\Omega)$.
For AdS_3 $\longrightarrow$ d'Alembertian $\Box = \Xi(\Omega)$.
It has continuous spectrum $\sigma(\Delta) = [\rho^2, +\infty[$ with $\rho^2 > 0$.

The resolvent of $\Xi(\Omega)$

$$R_\Omega(z) = (\Xi(\Omega) - z)^{-1}$$

is a bounded operator on $L^2(X)$ depending holomorphically on $z \in \mathbb{C} \setminus \sigma(\Xi(\Omega))$, i.e.

$$\mathbb{C} \setminus \sigma(\Omega) \ni z \longrightarrow R_\Omega(z) \in \mathcal{B}(L^2(G,\tau)) .$$

is a holomorphic operator-valued function.

Problem 1: For general X , does R_Ω admit a meromorphic extension to a Riemann surface above $\mathbb{C}$?
If so: what are the poles? What are the residues?
$\hookrightarrow$ resonances

Problem 2: At each pole z_0, one get a map
$R(z_0) : C_c^\infty(X) \to C^\infty(X)$. What is the image of this map ?

Problem 3: Because Ω is in the center of $U(\mathfrak{g}_\mathbb{C})$, the space $R(z_0)(C_c^\infty(X))$. is G invariant. What is this representation of G?
$\hookrightarrow$ Residue representations

Resonances of the Casimir on symmetric space : Previous works

- Real hyperbolic n-space (1995): **Guillopé** and **Zworski**

Theorem

For $X = H^n(\mathbb{R})$ *and* $\Omega = \mathbb{C}$, *the resolvent* R_Δ *has:*

- ◇ *holomorphic extension, if* n *is odd*
- ◇ *meromorphic extension (with infinitely many poles) if* n *even.*

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- Computation of resonances and residue representations for hyperbolic spaces: **Miatello** and **Will** (2000) and with a different method **Hilgert** and **Pasquale** (2009)

Theorem

X = Riemannian symmetric space of (real) rank one. Then $\zeta \mapsto R_\Delta(\zeta^2)$ extend to

- *a holomorphic function on $\mathbb{C}$ $\rightsquigarrow$ no resonances, if $X = H^n(\mathbb{R})$ with n odd,*
- *a meromorphic continuation with infinitely many (simple) poles, otherwise.*

The residue representations are all finite dimensional and irreducible.

- Meromorphic continuation on Riemannian symmetric spaces (2005): **Strohmaier**, **Mazzeo** and **Vasy**

Theorem

X = arbitrary Riemannian symmetric space of the noncompact type. There are $\Omega \subsetneq \mathbb{C}$ open with $\sigma(\Delta) \subset \Omega$ and M Riemann surface above Ω such that

$$R_\Delta : \Omega \backslash \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$$

admits holomorphic extension to M.

Problem: Ω is not large enough to find resonances.

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Resonances in higher rank: Hilgert, Pasquale, Przedinda for $G = \mathrm{SL}(3, \mathbb{R})$, BC_2, C_2 and when X is the product of 2 rank one Riemannian symmetric spaces of non-compact type.

Resonances on $\mathrm{U}(p, q; \mathbb{F}) / \mathrm{U}(1, \mathbb{F}) \times \mathrm{U}(p-1, q, \mathbb{F})$ has been given by Frahm and Polyxeni (2023). (where $\mathbb{F}$ is either the field of real, complex, quaternionic or octonionic numbers)

Define on $\mathbb{R}^4$:

$$\langle x, y\rangle := x_1 y_1 + x_2 y_2 - x_3 y_3 - x_4 y_4 \ .$$

Then

$$X = \{x \in \mathbb{R}^4 \mid \langle x, x\rangle = 1\} \ .$$

We use the fact $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,R) \times \mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}_{\mathrm{e}}(2,1) \simeq \mathrm{SL}(2,\mathbb{R})$.

V	$\mathbb{R}^4$	$\mathrm{Mat}_2(\mathbb{R})$
G	$\mathrm{SO}_{\mathrm{e}}(2,2)$	$\mathrm{SL}(2,R) \times \mathrm{SL}(2,\mathbb{R})/\pm \mathrm{Id}$
H	$\mathrm{SO}_{\mathrm{e}}(2,1)$	$\mathrm{SL}(2,\mathbb{R})$
	X	$\mathrm{SL}(2,\mathbb{R})$
$g \in G$, $v \in V$	$g \cdot v$	$g_1 \cdot v \cdot g_2$
	$\square$	$R(\Omega_{\mathrm{SL}(2,\mathbb{R})})$

So the resonances of $\square$ are those of the Laplacian on $\mathrm{SL}(2,\mathbb{R})$, through this isomorphism.

But the residue representations are not the ones of $\mathrm{SL}(2,\mathbb{R})$ because the spaces are invariant under the action of $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,R) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem

$$\left(R(\zeta^2+1)f\right)(g) = H(\zeta,f)(g) + \sum_{\substack{l\in\mathbb{Z}_+ \\ l<y}} \frac{H_l(f)(g)}{il+\zeta} .$$

The above formula give a meromorphic continuation of the resolvent of the positive Laplace operator on the half-plane above $\mathbb{R} - iy$ with simple poles at $\zeta = -il$ where $l \in \mathbb{Z}_+$, $l < y$. We make $y \to +\infty$. Let $S = \{(z,\zeta) \in \mathbb{C}^2 \mid z = \zeta^2 + 1\}$. Then the resolvent extends meromorphically from $S^+ = \{(z,\zeta) \in M \mid \Im(\zeta) > 0\}$ to S and with simple poles at $(-l+1, -il)$, $l \in \mathbb{Z}_+$. These poles are the resonances of the Laplace operator.

We define the residue representations $\mathscr{E}_l$ to be the image of the operator

$$H_l : C_c^\infty(X) \longrightarrow C_c^\infty(X)$$

Theorem

The residue representations $\mathscr{E}_l$ of the group $\mathrm{SO_e}(2,2)$ arising from the resonances of the d'Alembertian, acting on $C_c^\infty(X)$ is the sum of three irreducible unitarisable representations if $l \neq 0$. One is finite dimensional. The other two are infinite dimensional and mutually contragredient. The representation $\mathscr{E}_0$ is the sum of the two infinite dimensional representations given for $l = 0$.

The Langlands parameters of all of them are also given.
One big difference with the Riemannian case, also seen in the work of Frahm and Polyxeni, is that the representations are not irreducible and not all finite dimensional.