Discrete series representations of rank one semisimple Lie groups

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Irreducible = cannot be split into 2 representations
Unitary = $\pi(g) = \pi(g)^*$ for every $g \in G$

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Theorem (Harish-Chandra, Mautner, Segal...)

There exists a measure μ (Plancherel measure) on $\hat{G}$ such that

$$L^2(G) = \int_{\hat{G}} V_\pi \otimes V_{\overline{\pi}} d\mu(\pi) = \int_{\hat{G}_{cont}} V_\pi \otimes V_{\overline{\pi}} d\mu(\pi) + \sum_{\hat{G}_{discr}} V_\pi \otimes V_{\overline{\pi}} \mu(\pi)$$

Proposition

If $[(\pi, V_\pi)] \in \hat{G}$, then the following conditions are equivalent:

1. *The Plancherel measure assigns strictly positive mass to the one point set $\{(\pi, V_\pi)\}$,*
2. *for some nonzero $u, v \in V_\pi$, $g \mapsto (\pi(g)u, v)$ is in $L^2(G)$,*
3. *for every nonzero $u, v \in V_\pi$, $g \mapsto (\pi(g)u, v)$ is in $L^2(G)$*
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For $\mathrm{SL}(2,\mathbb{R})$ one Cartan subgroup is $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}\right\}$,

Maximal Compact $= \mathrm{SO}(2) = \left\{\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\right\}$

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Theorem (Harish-Chandra)

G has discrete series representations $\Leftrightarrow \mathrm{Rank}(G) = \mathrm{Rank}(K)$

Realizations of discrete series

- Narasimhan and Okamoto (1970) : If G/K is Hermitian symmetric, construction of most of the discrete series for G on spaces of square integrable harmonic forms of type $(0, q)$,
- Schmid (1971) : construction of most of the discrete series for G on "L^2-cohomology" groups of holomorphic line bundles on G/T, where T is a compact Cartan subgroup of G,
- Hotta (1972) : construction of most of the discrete series for G on certain eigenspaces of the Casimir of G, in the space of L^2-sections of some vector bundle,
- Parthasarathy (1972) : construction of most of the discrete series, studying the kernel of the Dirac operator,
- Flensted-Jensen (1980) : construction of some the discrete series of G/H, extend the result on the rank to G/H, for H subgroup of G given by an involution.

Framework

G: non-compact semisimple Lie group, finite center, $\mathrm{Lie}(G) = \mathfrak{g}$
K: maximal compact of G, $\mathrm{Lie}(K) = \mathfrak{k}$, so $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$
$A = \exp(\mathfrak{a})$: abelian group
$X := G/K$: Riemannian symmetric space of non-compact type
$M := Z_K(A)$, $\mathrm{Lie}(M) = \mathfrak{m}$

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Example:

$G = \mathrm{SO}_e(1, n)$
$K = \mathrm{SO}(n)$
$M = \mathrm{SO}(n-1)$

$$A = \left\{ \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{n-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$$

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$$H^n(\mathbb{R}) = G/K = \{x \in \mathbb{R}^{n+1} \mid x_1^2 - x_2^2 - \cdots - x_{n+1}^2 = 1\}$$

Real hyperbolic space

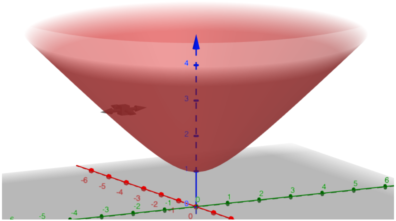

Figure: $H^2(\mathbb{R})$

We take an (homogeneous) vector bundle over G/K.
(τ, V_τ) := irreducible unitary representation of K which defines uniquely a homogeneous vector bundle

$$E_\tau := G \times V_\tau/_\sim \ ,$$

where $\big(g, v\big) \sim \big(gk, \tau(k^{-1})v\big)$.

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- $E_{\mathrm{triv}_K} \simeq X \times \mathbb{C} \quad \rightarrow$ generalization of the Riemannian symmetric spaces
- $G = \mathrm{SO}_e(1, n)$, $K = \mathrm{SO}(n)$ and $\tau_p := \Lambda^p Ad^*$ where Ad^* denote the coadjoint representation of K on $\mathfrak{p}_{\mathbb{C}}^*$,
  $\rightarrow \; E_{\tau_p} = \Lambda^p H^n(\mathbb{R}) := \Lambda^p \left(T_{\mathbb{C}}^*(H^n(\mathbb{R}))\right)$.
- $G = \mathrm{SL}(2, \mathbb{R})$: Representations τ of K are one dimensional, and thus E_τ is composed of the line bundles over $H^2(\mathbb{R})$.

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The smooth sections of E_τ can be identified with

$$C^\infty(G, \tau) := \left\{ f : G \xrightarrow{\text{smooth}} V_\tau \mid f(gk) = \tau(k^{-1}) f(g) \right\} .$$

$\mathbb{D}(E_\tau)$: set of homogeneous differential operators acting on smooth sections, i.e.

$$L(g)\ D = D\ L(g) \quad \text{for all } D \in D(E_\tau) \text{ and } g \in G$$

$$\Xi : U(\mathfrak{g}_{\mathbb{C}})^K \to \mathbb{D}(E_\tau)$$

$$X_1 \cdots X_n \mapsto \left(f \mapsto \frac{d}{dt_1} \cdots \frac{d}{dt_n} f\left(x \exp(t_1 X_1) \cdots \exp(t_n X_n)\right)\Big|_{t_i=0} \right)$$

(positive) Laplacian $\Delta = \Xi(-\Omega)$ where Ω is the Casimir operator

$$\Omega := -\sum X_i^2 + \sum Y_i^2 ,$$

where $\{X_i\}_{i=1,...,\dim \mathfrak{k}}$ and $\{Y_i\}_{i=1,...,\dim \mathfrak{p}}$ are othonormal basis of $\mathfrak{k}$ and $\mathfrak{p}$

It has continuous spectrum $\sigma(\Delta) = [\rho_\tau^2, +\infty[$ with $\rho_\tau^2 > 0$.

The resolvent of Δ

$$R_\Delta(z) = (\Delta - z)^{-1}$$

is a bounded operator on $L^2(G,\tau)$ depending holomorphically on $z \in \mathbb{C}\backslash\sigma(\Delta)$, i.e.

$$\mathbb{C}\backslash\sigma(\Delta) \ni z \longrightarrow R_\Delta(z) = (\Delta - z)^{-1} \in \mathcal{B}(L^2(G,\tau)) .$$

is a holomorphic operator-valued function.

Harmonic analysis on vector bundles

$M := Z_K(A)$, $\mathrm{Lie}(M) = \mathfrak{m}$

For a fixed representation $\tau \in \hat{K}$:

$$\hat{M}(\tau) := \{\sigma \in \hat{M} \mid \tau|_M \supset \sigma\}$$

For $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we denote by

$$\pi^\sigma_\lambda = \mathrm{Ind}\,^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes 1)$$

Harish-Chandra's notation for this induced representation is $\pi^\sigma_{i\lambda}$

the representation of the principal series, acting on the L^2 space $\mathscr{H}^\sigma_\lambda$.

Frobenius reciprocity theorem: $m(\sigma, \tau|_M) = m(\tau, \pi^\sigma_\lambda) =: m_{\sigma,\tau}$

Generalized spherical function $\varphi^{\sigma,\lambda}_\tau : G \to \mathrm{End}(V_\tau)$ defined by

$$\varphi^{\sigma,\lambda}_\tau(x) = \sum_{i=1}^{m_{\sigma,\tau}} P_i\ \pi^\sigma_\lambda(x)\ P_i^*$$

where P_i is the projection on the i-th component of τ in $\mathscr{H}^\sigma_\lambda$.

Rank one

Assumption: G is of real rank one $\iff$ $\dim \mathfrak{a} = 1$

G	K	$X = G/K$
$\mathrm{Spin}(n,1)$	$\mathrm{Spin}(n)$	real h. s.
$\mathrm{SU}(n,1)$	$\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$	complex h. s.
$\mathrm{Sp}(n,1)$	$\mathrm{Sp}(n)$	quaternionic h. s.
$\mathscr{F}_4$	$\mathrm{Spin}(9)$	octonionic h. p.

Facts :

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1. find the resonances of Δ which are the (simple) poles of the meromorphic continuation of R_Δ (on a Riemannian surface over $\mathbb{C}$)
2. Study the residue representations which arises from the resonances, namely

$$\mathscr{E}_k^\sigma := \{\varphi_\tau^{\sigma,\lambda_k} * f \mid f \in \Gamma^\infty(G,\tau)\} \curvearrowleft^L G$$

where $[\sigma] \in \hat{M}(\tau)$, $\lambda_k \sim$ resonance.

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Some of $\mathscr{E}_k^\sigma$ them are discrete series
AND
"G/K admits resonances" $\Leftrightarrow$ "$\mathrm{Rank}(G) = \mathrm{Rank}(K)$" in the real rank one case

$\rightsquigarrow$ Could we realized every discrete series r. as a residue representation ? The answer is yes for $G = \mathrm{SL}(2,\mathbb{R})$ (C. Will in 2003)

A case by case proof show that:

Theorem

For $G = \mathrm{Spin}(2n, 1)$ and $\mathrm{SU}(n, 1)$, every discrete series representation can be realize as a residue representation. Concretely :

- *if $G = \mathrm{Spin}(2n, 1)$, the discrete series of H-C parameter Λ, is realised by the residue representations $\mathscr{E}_\tau^{\sigma,\lambda}$, where*
  τ has highest weight $\Lambda - \rho + 2\rho_n$,
  $\sigma \in \hat{M}(\tau)$ arbitrary and
  λ is the first non-zero pole of the Plancherel density
- *if $G = \mathrm{SU}(n, 1)$, the discrete series of H-C parameter Λ, is realised by the residue representations $\mathscr{E}_\tau^{\sigma,\lambda}$, where*
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A similar proof should work for the others cases of real rank one, but I'm searching a generic proof (which can maybe be extended to real rank one greater than 1).

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This result should be true for all the real ranked one groups.

Thank you for your attention ! Do you have any questions ?