DISCRETE SERIES REPRESENTATIONS AS RESIDUE REPRESENTATIONS OF THE LAPLACE OPERATOR

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Contents

Abstract. Various realizations of the representations belonging to the discrete series has been carried out. In this paper, we use the realization given by Knapp and Wallach in [KW76, KW80] to show that, for the real rank one semisimple Lie groups, every discrete series representation can be seen as a residue representation of the Laplace operator acting on sections of some homogeneous vector bundle.

1. Introduction

Discrete series representations have been introduced by Harish-Chandra in [HC65, HC66]. Since then, various realizations of these representations have been carried out. Here are some well known references

- Narasimhan and Okamoto [NO70]: construction of most of the discrete series for G on spaces of square integrable harmonic forms of type $(0, q)$, if G/K is Hermitian symmetric;
- Atiyah, Schmid [Sch71, Sch76, AS77] : construction of most of the discrete series for G on L^2-cohomology groups of holomorphic line bundles on G/T, where T is a compact Cartan subgroup of G;
- Hotta [Hot71]: construction of most of the discrete series for G on certain eigenspaces of the Casimir acting on the space of L^2-sections of some vector bundle;
- Parthasarathy [Par72]: construction of most of the discrete series, studying the kernel of the Dirac operator.

There is also a realization carried out by Knapp and Wallach [KW76, KW80], viewing discrete series representations as the image of the Szegö map. We are using this result in the present article.

Let G be a connected semisimple Lie group with finite center, and with Iwasawa decomposition $G = KAN$. As shown in [HC65], G has discrete series representations if and only if it has a

2010 *Mathematics Subject Classification.* Primary: 22E45, 20G05, 22D10 ; secondary: 43A85,58J50.

compact Cartan subgroup $T \subset K$. We consider, for an arbitrary irreducible representation τ of K, the sections of the homogeneous vector bundle $E_\tau := G \times_\tau K$. Let $\mathfrak{a}$ be the Lie algebra of A and let M be the centralizer of $\mathfrak{a}$ in K. Then, for an irreducible representation σ of M and $\lambda \in \mathfrak{a}_\mathbb{C}^*$, we define the space $\mathscr{E}_\tau^{\sigma,\lambda}$ (see (17)) of smooth sections convoluted with the generalized spherical function $\varphi_\tau^{\sigma,\lambda}$ (see Definition 3.1). The first step is to show that the discrete series representations are infinitesimally equivalent with the left action of $\mathfrak{g}$ on the K-finite vectors of $\mathscr{E}_\tau^{\sigma,\lambda}$ (See Corollary 4.1).
When G is of real rank one, i.e. when $\dim \mathfrak{a} = 1$, it has been [Rob22], that given a resonance of the Laplace operator acting on smooth sections E_τ, one constructs a corresponding representation of G, called a *resonance representation*. This representation is exactly $\mathscr{E}_\tau^{\sigma,\lambda}$, with $\lambda \in \mathfrak{a}_\mathbb{C}^*$ a pole of the Plancherel density p_σ of σ. In [Rob22], we saw that some of them are discrete series representations. Previously, it has been shown in [Wil03] that all the discrete series representations can be realized this way, if $G = \mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$. The main result of this paper is the following.

Theorem 1
Suppose G is of real rank one. Every discrete series representation of G can be realized as a residue representation of the Laplace operator for some homogeneous vector bundle. Namely, if π_Λ is the discrete series with Harish-Chandra parameter Λ, then it is realized as the residue representation $\mathscr{E}_{\tau_\Lambda}^{\sigma_\Lambda,\lambda_\Lambda}$, which parameters are defined in Theorem 2 below.

Explicitly, τ_Λ is the lowest K-type of π_Λ, σ_Λ is the representation of M generated by a highest weight vector in the representation space of V_{τ_Λ} and λ_Λ is defined in (16).
In Section 2 we introduce the notations and the context of this paper. In Section 3 we collect some definitions and facts concerning the vector-valued Fourier transform, its relations with the generalized spherical functions, and the Eisenstein integral formulas. In Section 4, we recall the main theorem of [KW76] and we relate it, in Corollary 4.1 to the spaces $\mathscr{E}_\tau^{\sigma,\lambda}$ defined in (17). Finally, in section 5, we introduce the residue representations of the Laplace operator of real rank one groups, and we prove Theorem 1. Section 6 is devoted to a case by case analysis, for $G = \mathrm{SO}(2n,1)$ and $G = \mathrm{SU}(n,1)$, using the structure of the principal series in [Col85] and similar methods as in [Rob23].

2. Notations and context

We shall use the standard notations $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the nonnegative integers, the integers, the real numbers, the complex numbers.
For every Lie group, we will use the corresponding gothic letter for its Lie algebra, the subscript $_\mathbb{C}$ for the complexification and the exponent * for the algebraic dual. The complex linear forms of the Lie algebra of a Lie group H is then denoted by $\mathfrak{h}_\mathbb{C}^*$.
For every group Γ, we denote by $\hat{\Gamma}$ be the set of all equivalence classes of irreducible unitary representations of Γ.

Context. Let G be a connected non-compact real semisimple Lie group with finite center and let $B(\cdot,\cdot)$ be the Killing form of the Lie algebra $\mathfrak{g}$ of G. We denote by θ a Cartan involution of $\mathfrak{g}$, by $\mathfrak{k}$ the set of fixed points of θ and by $\mathfrak{p}$ the eigenspace of θ for the eigenvalue -1. In other words:

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta X = X\} \quad \text{and} \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \theta X = -X\} \, .$$

The Cartan decomposition of the Lie algebra $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. The corresponding connected Lie subgroup K of G is maximal compact.
Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $A = \exp \mathfrak{a}$ be its associated Lie subgroup of G. The exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism between $\mathfrak{a}$ and A. The inverse map is the logarithm written log in the following.
Let M be the centralizer of $\mathfrak{a}$ in K. Define first the noncompact Cartan Lie algebra and the root system associated. Let $\mathfrak{h}^-$ be a Cartan Lie algebra of $\mathfrak{m}$. Then $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{h}^-$ is a (non compact) Cartan subalgebra of $\mathfrak{g}$. Let Π be the set of roots of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. It consists of all nonzero linear forms $e \in \mathfrak{h}_\mathbb{C}^*$ for which the vector space

$$\mathfrak{g}^e := \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = e(H)X \text{ for every } H \in \mathfrak{h}_\mathbb{C}\}$$

contains nonzero elements. We choose a set Π^+ of positive roots in Π. Let $\Pi_\mathfrak{m}$ (respectively $\Pi_\mathfrak{m}^+$) be the set of (positive) roots of the pair $(\mathfrak{m}_\mathbb{C}, \mathfrak{h}_\mathbb{C}^-)$. Let

$$\Sigma := \{e|_\mathfrak{a} \mid e \in \Pi\} \setminus \{0\}$$

be the set of restricted roots. It coincides with the roots of the pair $(\mathfrak{g}, \mathfrak{a})$. Let Σ_+ be the set of positive restricted roots, i.e. the set $\{e|_\mathfrak{a} \mid e \in \Pi^+\} \setminus \{0\}$. Let $\rho_\mathfrak{a} := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ be the half sum of the positive restricted roots counted with their multiplicities. Set $\mathfrak{n} = \bigoplus_{e|_\mathfrak{a} \in \Sigma^+} \mathfrak{g}^e \cap \mathfrak{g}$ and let $N = \exp \mathfrak{n}$ the connected Lie subgroup of G having $\mathfrak{n}$ for Lie algebra. Finally, we denote the respective half sums of positive roots by ρ, $\rho_\mathfrak{m}$ and $\rho_\mathfrak{a}$ for the restricted roots.
Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ such that $\mathfrak{h}^- \subset \mathfrak{t}$. Then $\mathfrak{t}$ is a (compact) Cartan subalgebra of $\mathfrak{g}$, because we supposed that $\mathfrak{g}$ has a compact Cartan. The set Δ of roots of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ consists of all nonzero linear forms $\varepsilon \in \mathfrak{t}_\mathbb{C}^*$ for which the vector space

$$\mathfrak{g}_\varepsilon := \{X \in \mathfrak{g}_\mathbb{C} \mid [H, X] = \varepsilon(H)X \text{ for every } H \in \mathfrak{t}_\mathbb{C}\}$$

contains nonzero elements.
We choose a set Δ^+ of positive roots in Δ which is compatible with Π_+, i.e. such that a root $\varepsilon \in \Delta$ is positive when $\varepsilon|_{\mathfrak{h}_\mathbb{C}^-} \in \Pi_\mathfrak{m}^+$. Let also $\Delta_\mathfrak{k}$ (respectively $\Delta_\mathfrak{k}^+$) be the set of (positive) roots of the pair $(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ known as compact roots and $\Delta_n = \Delta \setminus \Delta_\mathfrak{k}$ (respectively $\Delta_n^+ = \Delta^+ \setminus \Delta_\mathfrak{k}^+$) be the set of (positive) non-compact roots. Finally, we denote the respective half sums of positive roots by δ, $\delta_\mathfrak{k}$, δ_n.
For any root $\varepsilon \in \Delta$, there is a choice of root vectors $E_\varepsilon \in \mathfrak{g}_\varepsilon$ such that

$$B(E_\varepsilon, E_{-\varepsilon}) = \frac{2}{\langle \varepsilon, \varepsilon \rangle} \quad \text{and} \quad \theta(\overline{E_\varepsilon}) = -E_{-\varepsilon} \ ,$$

where $\overline{\cdot}$ is the usual complex conjugation. Set $H_\varepsilon = [E_\varepsilon, E_{-\varepsilon}]$. Then $\varepsilon(H_\varepsilon) = 2$ and

$$\begin{array}{ll} E_\varepsilon + E_{-\varepsilon} \text{ and } i(E_\varepsilon - E_{-\varepsilon}) & \text{are in } \mathfrak{g} \text{ if } \varepsilon \text{ is noncompact,} \\ E_\varepsilon - E_{-\varepsilon} \text{ and } i(E_\varepsilon + E_{-\varepsilon}) & \text{are in } \mathfrak{g} \text{ if } \varepsilon \text{ is compact.} \end{array}$$

Definition 2.1 (Definition 4.1 in [KW76]**)**
A sequence $\varepsilon_1, ..., \varepsilon_m$ of positive noncompact roots is a *fundamental sequence* if

(1) the ε_j form a strongly orthogonal set, i.e. no $\varepsilon_i \pm \varepsilon_j$ is a root,
(2) $\mathfrak{a}_\varepsilon = \sum_{j=1}^m \mathbb{R}(E_{\varepsilon_j} + E_{-\varepsilon_j})$ is maximal abelian in $\mathfrak{p}$,
(3) ε_j is a simple root in the subsystem of roots strongly orthogonal to $\varepsilon_1, ..., \varepsilon_{j-1}$,

(4) for each $\gamma \in \Delta_n^+$ either
   (a) $|\varepsilon(\gamma)| \geq |\gamma|$, or
   (b) $|\varepsilon(\gamma)| < |\gamma|$ and $\gamma - 3\varepsilon(\gamma)$ is a root,

where $\varepsilon(\gamma)$ is the first ε_j such that γ is not strongly orthogonal to ε_j.

If G is of real rank one, the fundamental sequence is one noncompact root ε and then $\mathfrak{a}_\varepsilon := \mathbb{R}(E_\varepsilon + E_{-\varepsilon})$ is maximal abelian in $\mathfrak{p}$. As we defined $\mathfrak{h}$ and $\mathfrak{t}$ compatible, one can choose the positive non compact root in $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, such that the Cayley transform defined from ϵ, send $\mathfrak{t}_\mathbb{C}$ onto $\mathfrak{h}_\mathbb{C}$. Thus $\mathfrak{a}_\varepsilon = \mathfrak{a}$.

3. Vector valued Helgason-Fourier transform and generalized spherical functions

In this section we review some basic facts on generalized spherical functions and Camporesi's extension of the Helgason-Fourier transform to homogeneous vector bundles. We refer the reader to [Cam97] for more information.

Let (τ, V_τ) be an irreducible unitary representation of K and let $E_\tau := G/K \times_\tau V_\tau$ denote the homogeneous vector bundle over G/K. For the definition and properties of E_τ, we refer the reader to [Wal73, §5.2, p. 114]. We write $\Gamma^\infty(E_\tau)$ for the space of all smooth sections of E_τ. As proved in [Wal73, §5.4, p. 119], there is a vector-space isomorphism between $\Gamma^\infty(E_\tau)$ and

$$C^\infty(G,\tau) := \{f : G \to V_\tau \text{ smooth} \mid f(xk) = \tau(k^{-1})f(x) \ \text{ for all } x \in G \text{ and } k \in K\} \ .$$

Let

$$m(\tau|_M, \sigma) := \dim \operatorname{Hom}_M(V_\tau, V_\sigma) \ . \tag{1}$$

and

$$\hat{M}(\tau) := \{\sigma \in \hat{M} \mid m(\sigma, \tau|_M) \geq 1\}$$

denote the set of unitary irreducible representations of M which occur in the restriction of τ to M. We will denote by d_γ the dimension of a representation γ. For $\sigma \in \hat{M}(\tau)$, let χ_σ denote its character and let

$$P_\sigma = d_\sigma \int_M \tau(m^{-1})\chi_\sigma(m)\, dm \, , \tag{2}$$

be the projection of V_τ onto its σ-isotypic component. According to the Iwasawa decomposition $G = KAN$, every element x in G can be uniquely written as

$$x = \mathbf{k}(x)\mathbf{a}(x)\mathbf{n}(x) = \mathbf{k}(x)e^{\mathbf{H}(x)}\mathbf{n}(x) \ , \tag{3}$$

where $\mathbf{k}(x) \in K$, $\mathbf{a}(x) \in A$, $\mathbf{H}(x) \in \mathfrak{a}$ and $\mathbf{n}(x) \in N$. In the following, we set

$$a^\lambda := e^{\lambda(\log a)} \ \text{ for } a \in A \text{ and } \lambda \in \mathfrak{a}_\mathbb{C}^* \ . \tag{4}$$

Let $C_c^\infty(G,\tau)$ be the space of compactly supported functions in $C^\infty(G,\tau)$. According to [Cam97, Theorem 1.1], the vector-valued Helgason-Fourier transform of $f \in C_c^\infty(G,\tau)$ is the function from $\mathfrak{a}_\mathbb{C}^* \times K$ to the Hilbert space V_τ defined by

$$\widetilde{f}(\lambda, k) := \int_G F^{i\overline{\lambda}-\rho}(x^{-1}k)^* f(x)\ dx \ . \tag{5}$$

Here, for $\mu \in \mathfrak{a}_\mathbb{C}$ and $x \in G$,

$$F^\mu(x) := e^{\mu(\mathbf{H}(x))}\tau(\mathbf{k}(x)) \tag{6}$$

and * denotes the V_τ-Hilbert space adjoint.

For $(\sigma, V_\sigma) \in \hat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, we denote by

$$\pi^\sigma_\lambda = \operatorname{Ind}{}^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes \mathrm{triv})$$

the representation to G induced from the representation $\sigma \otimes e^{i\lambda} \otimes \mathrm{triv}$ of MAN. We also denote by $\mathscr{H}^\sigma_\lambda$ its representation space. These are the principal series representations.
Their compact picture is obtained by restriction of the elements of $\mathscr{H}^\sigma_\lambda$ to K. Their representation space does not depend on λ and we denote it by $\mathscr{H}^\sigma$. It is the Hilbert completion of

$$\{f : K \to \mathscr{H}_\sigma \mid f(km) = \sigma(m^{-1})f(k) \ \text{ for all } k \in K,\ m \in M\}$$

with respect to L^2 inner product. The G-action is given by

$$\pi^\sigma_\lambda(g)f(k) := e^{-(i\lambda+\rho)\mathbf{H}(g^{-1}k)} f(\mathbf{k}(g^{-1}k))$$

for all $g \in G$, $k \in K$ and $f \in \mathscr{H}^\sigma$. The representation π^σ_λ is unitary for $\lambda \in \mathfrak{a}^*$.

Remark:
One may point out that we are not using the usual notations. For instance in [Vog81], one often finds $(\pi^\sigma_{i\lambda}, \mathscr{H}^\sigma_{i\lambda})$, instead of our $(\pi^\sigma_\lambda, \mathscr{H}^\sigma_\lambda)$ for the representation $\operatorname{Ind}{}^G_{MAN}(\sigma \otimes e^{i\lambda} \otimes \mathrm{triv})$. In the above definitions, we are following the notation of [Hel08, chapter VI, §3] and [Cam97]. In our notations, the $(\pi^\sigma_\lambda, \mathscr{H}^\sigma_\lambda)$ belongs to the *unitary principal series* if and only if $\lambda \in \mathfrak{a}^*$.

Let P_τ denote the projection of $\mathscr{H}^\sigma$ onto its subspace of vectors which transform under K according to τ, that is,

$$P_\tau := d_\tau \int_K \pi^\sigma_\lambda(k)\chi_\tau(k^{-1})\, dk\,. \tag{7}$$

Definition 3.1
The spherical function $\varphi^{\sigma,\lambda}_\tau$ is defined as the $\operatorname{End}(V_\tau)$-valued function on G given by

$$\varphi^{\sigma,\lambda}_\tau(x) := \varphi^{\pi^\sigma_\lambda}_\tau(x) := d_\tau \int_K \tau(k)\psi^{\sigma,\lambda}_\tau(xk^{-1})\, dk\,, \tag{8}$$

where

$$\psi^{\sigma,\lambda}_\tau(x) := \operatorname{Tr}\left(P_\tau \pi^\sigma_\lambda(x) P_\tau\right). \tag{9}$$

Let $\operatorname{Hom}_K(\mathscr{H}^\sigma_\lambda, V_\tau)$ be the space of K-intertwining operators between $\pi^\sigma_\lambda|_K$ and τ. We equip this space with the scalar product $\langle P, Q\rangle := \frac{1}{d_\tau}\operatorname{Tr}(PQ^*)$, where $*$ denotes the adjoint. We fix an orthonormal basis $\{P_\xi\}_{\xi=1,\ldots,m(\tau|_M,\sigma)}$ of this space. Then

$$\varphi^{\sigma,\lambda}_\tau(g) = \sum_{\xi=1}^{m(\tau|_M,\sigma)} P_\xi\ \pi^\sigma_\lambda(g)\ P^*_\xi\,. \tag{10}$$

See pp. 268–269 and 273 in [Cam97].
The spherical function $\varphi^{\sigma,\lambda}_\tau$ can also be described as an Eisenstein integral (see [Cam97, Lemma 3.2]),

$$\varphi^{\sigma,\lambda}_\tau(x) = \frac{d_\tau}{d_\sigma} \int_K \tau(\mathbf{k}(xk))\ P_\sigma\ \tau(k^{-1})\ e^{(i\lambda-\rho)(\mathbf{H}(xk))}\ dk\ . \tag{11}$$

Notice that $\varphi_\tau^{\sigma,\lambda}$ satisfies $\varphi_\tau^{\sigma,\lambda}(k_1xk_2) = \tau(k_1)\varphi_\tau^{\sigma,\lambda}(x)\tau(k_2)$ for every $x \in G$ and $k_1, k_2 \in K$. The convolution with a function $f \in C_c^\infty(G,\tau)$ is defined by:

$$(\varphi_\tau^{\sigma,\lambda} * f)(x) := \frac{d_\tau}{d_\sigma}\int_G \varphi_\tau^{\sigma,\lambda}(x^{-1}g)f(g)\ dg\ . \tag{12}$$

It can be expressed in terms of the vector-valued Helgason-Fourier transform of f, according to [Cam97, Proposition 3.3],

$$(\varphi_\tau^{\sigma,\lambda} * f)(x) = \frac{d_\tau}{d_\sigma}\int_K F^{i\lambda-\rho}(x^{-1}k)\ P_\sigma\ \widetilde{f}(\lambda,k)\ dk\ . \tag{13}$$

More information about these generalized spherical functions can be found in [Cam97, Rob22].

4. Map into the Szegö kernel

Let $\tau \in \hat{K}$, $\sigma \in \hat{M}(\tau)$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. In [KW76], the authors introduce the map $S_\lambda^\sigma : \mathscr{H}^\sigma \to C^\infty(G,\tau)$ defined, for all $\phi \in \mathscr{H}^\sigma$ and $x \in G$ by [1]

$$S_\lambda^\sigma(\phi)(x) = \int_K \tau(k)\pi_\lambda^\sigma(x)\phi(k)\ dk = \int_K F^{i\lambda-\rho}(x^{-1}k)\phi(k)\ dk\ . \tag{14}$$

In [HC65, Theorem 2], Harish-Chandra associated to any nonsingular integral form $\Lambda \in \mathfrak{t}_\mathbb{C}^*$ an invariant eigendistribution Θ_Λ. In [HC66], he proved the existence of a discrete series representation π_Λ, with character Θ_Λ. Every discrete series representation is given this way.

Theorem 2 (Theorem A in [KW80]**)**
Let $\Lambda \in \mathfrak{t}_\mathbb{C}^*$ non singular and dominant with respect to Δ^+ and let $\mu_\Lambda = \Lambda + \rho_n - \rho_\mathfrak{k}$. Let $(\tau_\Lambda, V_{\tau_\Lambda}) \in \hat{K}$ be of highest weight μ_Λ and let v_Λ be a nonzero highest weight vector in V_{τ_Λ}. Define $\sigma_\Lambda \in \hat{M}(\tau)$ by restricting $\tau_\Lambda(M)$ to the M-cyclic subspace generated by v_Λ. Let

$$\sigma_\Lambda = \sum_{i=1}^r \sigma_j \tag{15}$$

be the decomposition of σ_Λ into irreducible representations of M. Define $\lambda_\Lambda \in \mathfrak{a}_\mathbb{C}^*$ by

$$\lambda_\Lambda(E_{\alpha_j} + E_{-\alpha_j}) := 2i\frac{\langle\Lambda,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \tag{16}$$

The operators $S_{\lambda_\Lambda}^{\sigma_j}$ carry the $(\mathfrak{g},K)$-module $(\pi_{\lambda_\Lambda}^{\sigma_\Lambda}, \mathscr{H}^{\sigma_\Lambda})$ onto the discrete series π_Λ.

When G is of real rank one, $(\sigma_\Lambda, H_\lambda)$ is an irreducible representation of M, i.e. $r = 1$. See [KW80, Final remark].

> **Remark:**
>
> In [KW76, Theorem 1.1], the principal series in are defined through the parameter $\nu \in \mathfrak{a}_\mathbb{C}^*$ such that $\lambda_\Lambda = -i(\rho_\mathfrak{a} - \nu)$. Our definition of λ_Λ in (16) is consistent with the definition of ν in [KW76, (6.5)]. In fact, Let us fix fundamental sequence defined in Definition 2.1. Let

[1] The action of G is here on the left and in [KW76] is on the right, which changes the formula.

ε_j be a non compact root belonging to that fundamental sequence. We set m_j and n_j as the integers, depending on ε_j, defined by

$$m_j := \left| \{\gamma \in \Delta_n^+ \mid \varepsilon(\gamma) = \varepsilon_j \text{ and } -\varepsilon(\gamma) + \gamma \in \Delta\} \right| ,$$

$$n_j := \left| \{\gamma \in \Delta_n^+ \mid \varepsilon(\gamma) = \varepsilon_j \text{ and } \varepsilon(\gamma) + \gamma \in \Delta\} \right| .$$

Here $|Q|$ state for the cardinality of the set Q. Recall that for $\gamma \in \Delta_n^+$, the root $\varepsilon(\gamma)$ is defined in Definition 2.1. The parameter ν is defined by

$$\nu(E_{\varepsilon_j} + E_{-\varepsilon_j}) := \frac{2\langle \Lambda + \rho_n - \rho_{\mathfrak{k}} + n_j\varepsilon_j, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} \qquad \text{([KW76, (6.5)])}$$

In the following computation, we are using the following facts:

$$\begin{aligned} &\rho_{\mathfrak{a}}(E_{\varepsilon_j} + E_{-\varepsilon_j}) = 1 + m_j + n_j \ , \\ &\frac{2\langle \rho_{\mathfrak{k}} - \rho_n, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} = -1 - m_j + n_j \ . \end{aligned} \qquad \text{([KW76, 8.5])}$$

So we have

$$\begin{aligned} (\rho_{\mathfrak{a}} - \nu)(E_{\varepsilon_j} + E_{-\varepsilon_j}) &= \rho_{\mathfrak{a}}(E_{\varepsilon_j} + E_{-\varepsilon_j}) - \frac{2\langle \Lambda, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} - \frac{2\langle \rho_n - \rho_{\mathfrak{k}}, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} - 2n_j \\ &= 1 + m_j + n_j + \frac{2\langle \Lambda, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} - 1 - m_j + n_j - 2n_j \\ &= -\frac{2\langle \Lambda, \varepsilon_j\rangle}{\langle \varepsilon_j, \varepsilon_j\rangle} \ . \end{aligned}$$

And, this explains our definition (16)

We will use this theorem to understand the action of $(\mathfrak{g}, K)$ on

$$\mathscr{E}_\tau^{\sigma,\lambda} := \{\varphi_\tau^{\sigma,\lambda} * f \mid f \in C_c^\infty(G, \tau)\} \tag{17}$$

Formula (13) and (14) imply the following lemma.

Lemma 4.1
Up to a constant multiple, for all $f \in C_c^\infty(G,\tau)$,

$$\varphi_\tau^{\sigma,\lambda} * f = S_\lambda^\sigma\Big(P_\sigma \widetilde{f}(\lambda, \cdot)\Big) \ . \tag{18}$$

Lemma 4.2
The function $k \mapsto P_\sigma \widetilde{f}(\lambda,\cdot)$ lies in $\mathscr{H}^\sigma$. Moreover the operator

$$f \mapsto P_\sigma \widetilde{f}(\lambda, \cdot)$$

intertwines the left actions of G on $C_c^\infty(G,\tau)$ and $\mathscr{H}^\sigma$.

Proof. For all $m \in M$, we have $\widetilde{f}(\lambda, km) = \tau(m)^{-1}\widetilde{f}(\lambda, k)$, mainly because in the Iwasawa decomposition $\mathbf{k}(gm) = \mathbf{k}(g)m$ and $\mathbf{a}(gm) = \mathbf{a}(g)m$, for all $g \in G$. As P_σ intertwines σ and τ,

$P_\sigma \widetilde{f}(\lambda,\cdot) \in \mathscr{H}^\sigma$. We also have, for all $g \in G$ and $k \in K$,

$$\widetilde{\big(L(g)f\big)}(\lambda,k) = \int_G F^{i\bar{\lambda}-\rho}(x^{-1}g^{-1}k)^* f(x)\ dx\ .$$

Also, since

$$\begin{aligned} x^{-1}g^{-1}k &= x^{-1}\mathbf{k}(g^{-1}k)\mathbf{a}(g^{-1}k)\mathbf{n} = \mathbf{k}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)\mathbf{a}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)\mathbf{n}'\mathbf{a}(g^{-1}k)\mathbf{n} \\ &= \mathbf{k}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)\mathbf{a}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)\mathbf{a}(g^{-1}k)\mathbf{n}''\mathbf{n}, \end{aligned}$$

$\mathbf{k}(x^{-1}g^{-1}k) = \mathbf{k}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)$ and $\mathbf{a}(x^{-1}g^{-1}k) = \mathbf{a}\big(x^{-1}\mathbf{k}(g^{-1}k)\big)\mathbf{a}(g^{-1}k)$. Finally,

$$\widetilde{\big(L(g)f\big)}(\lambda,k) = \int_G \mathbf{a}(g^{-1}k)^{-i\lambda-\rho} F\big(x^{-1}\mathbf{k}(g^{-1}k)\big)^* f(x)\ dx\ = \mathbf{a}(g^{-1}k)^{-i\lambda-\rho}\widetilde{f}(\lambda,\mathbf{k}(g^{-1}k)).$$

So the map is intertwining.

■

An argument by Dirac distributions shows that the space $\mathscr{E}_\tau^{\sigma,\lambda}$ is nonzero. We have so:

Corollary 4.1
With data chosen as in Theorem 2, the representations $\mathscr{E}_{\tau_\Lambda}^{\sigma_j,\lambda_\Lambda}$ are infinitesimally equivalent to the discrete series representation π_Λ.

5. Residue representations

For G of real rank one, i.e. when the dimension of $\mathfrak{a}$ is 1, the resonances of the Laplace operator acting on sections of E_τ, $\tau \in \hat{K}$ arbitrary, have been found in [Rob22, Theorem 1]. Since the resonances of the Laplacian for G of higher real rank are not known, we are restricting ourself to the rank one case. The following statement follows from [Rob22, Theorem 1].

Lemma 5.1
Let $\sigma \in \hat{M}(\tau)$ and let λ_σ be a pole of the Plancherel density p_σ. Each resonance z of the Laplace operator acting on $C_c^\infty(G,\tau)$ belongs to a Riemannian surface covering $\mathbb{C}$ and can be written as the pair

$$(z,\zeta) = \big(-\lambda_\sigma^2|\alpha|^2 - \langle\rho_\mathfrak{a},\rho_\mathfrak{a}\rangle + \langle\mu_\sigma+\rho_\mathfrak{m},\mu_\sigma+\rho_m\rangle\ ,\lambda_\sigma|\alpha|\ \big)\ , \tag{19}$$

where $\zeta := \sqrt{z - \langle\rho_\mathfrak{a},\rho_\mathfrak{a}\rangle + \langle\mu_\sigma+\rho_\mathfrak{m},\mu_\sigma+\rho_m\rangle}$. Here $\sqrt{\cdot}$ denotes the single-valued branch of the square root function determined on $\mathbb{C}\setminus[0,+\infty[$ by the condition $\sqrt{-1}=-i$.

Because of Corollary 5.1 to prove Theorem 1, one has to show that the data $(\sigma_\Lambda,\lambda_\Lambda)$ defined in Theorem 2 are such that λ_Λ is a pole of the Plancherel density p_{σ_Λ}.
Let γ be a noncompact root, positive relative to Δ^+. Take E_γ as in Section 2. So $E_\gamma + E_{-\gamma} \in \mathfrak{p}$ and $\mathfrak{a} = \mathbb{R}(E_\gamma + E_{-\gamma})$. Let $u_\gamma := \exp\pi(E_\gamma+E_{-\gamma})/4$. The Cayley transform $\mathsf{Ad}u_\gamma : \mathfrak{t}_\mathbb{C} \to \mathfrak{h}_\mathbb{C}$ is such that $\Pi = \Delta\circ\mathsf{Ad}u_\gamma^{-1}$ and $\Pi_\mathfrak{m}$ is exactly $\{\beta = (\beta\circ\mathsf{Ad}u_\gamma^{-1} \mid (\beta,\gamma)=0\}$. The root $\gamma\circ\mathsf{Ad}u_\gamma^{-1}$ is the real root of Π^+.

Proposition 5.1 (Proposition 8.2 in [KW76]**)**
Let $s_0 \in W(\Pi_\mathfrak{m})$ such that $s_0\Pi_m^+ = -\Pi_m^+$. We have

$$\chi_{\sigma_\Lambda,\lambda_\Lambda} = s_0(\Lambda\circ\mathsf{Ad}(u_\gamma^{-1})) + \rho \tag{20}$$

Even if such an explicit formula is not known for an arbitrary group G, in the rank-one case, several authors have computed it explicitly ([Oka65], [Mia79]; see also [War72, Epilogue, pp. 414 ff.], [KS71, page 543]). The Plancherel density p_σ for rank one Lie groups is the product of two factors and depends on the highest weight of σ. The first one is a polynomial function in λ, denoted $\mathcal{P}_\sigma$. The second factor is either a hyperbolic tangent or a hyperbolic cotangent (or 1 if $G = \mathrm{Spin}(2n+1,1)$, but there is no discrete series for this group).
Let us first recall the precise formula of the polynomial part $\mathcal{P}_\sigma$ of the Plancherel density p_σ. This formula, for real rank one groups, can be found in [KS71, page 543]:

$$\mathcal{P}_\sigma(\lambda) = \prod_{\eta \in \Pi^+} \langle \eta, \chi_{\sigma,\lambda} \rangle \ ,$$

so here

$$\mathcal{P}_{\sigma_\Lambda}(\lambda_\Lambda) = \prod_{\eta \in \Pi^+} \langle \eta, s_0(\Lambda \circ \mathsf{Ad}(u_\gamma^{-1})) + \rho \rangle \ ,$$

But

$$\Big\langle \underbrace{\langle s_0\eta \circ \mathsf{Ad}(u_\gamma), \Lambda \rangle}_{\geq 0} + \underbrace{\langle \eta, \rho \rangle}_{>0} \Big\rangle 0$$

because Λ is Δ^+-dominant. So λ_Λ is not a zero of the polynomial part of the Plancherel density. The second part of p_{σ_Λ} is either a hyperbolic tangent or a hyperbolic cotangent. Because of [Rob23, Proposition 5.1], we know that λ_Λ is a pole of this part if and only the infinitesimal character $s_0(\Lambda \circ \mathsf{Ad}(u_\gamma^{-1})) + \rho$ is a regular infinitesimal character. This is the case, by definition of Λ. So λ_Λ is a pole of the Plancherel density p_{σ_Λ} and thus it induces a resonance of the Laplace operator and a residue representation $\mathscr{E}_{\tau_\Lambda}^{\sigma_\Lambda,\lambda_\Lambda}$, thanks to Corollary 5.1. This prove Theorem 1.

6. Case by case proof for $G = \mathrm{SO}(2n,1)$ or $G = \mathrm{SU}(n,1)$

In this section, for $G = \mathrm{SO}(2n,1)$ or $G = \mathrm{SU}(n,1)$, we are going to give a proof of Theorem 1 using the explicit description of representations of real rank one Lie groups given in [Col85]. This allows us to show that, for suitable $\tau \in \hat{K}$ and $\sigma \in \hat{M}(\tau)$, the discrete series representations can correspond to the pole λ of p_σ, which is the closest to 0. This result looks enough interesting for giving a proof of it and explains concretely how to compute τ and σ in that cases. The cases $G = \mathrm{Sp}(n,1)$ or $G = \mathrm{F}_4$ are also doable the same way, with much more computations. This might be interesting to see if it's also possible to keep λ as the closest pole to 0 in those cases.
We keep the notations of previous sections. We choose τ_λ to be the K-type with highest $\mu_\Lambda = \Lambda - \rho + 2\rho_n$ (minimal K-type of π_{σ_Λ}). We know that $G = \mathrm{SO}(2n,1)$ and $G = \mathrm{SU}(n,1)$ are multiplicity free cases (see for instance [Koo82]), so $m(\tau_\Lambda|_M, \sigma_\Lambda) = 1$. Because of [Rob22, Lemma 4.1], we know that the residue representation $\mathscr{E}_\lambda := \mathscr{E}_{\tau_\Lambda}^{\sigma_\Lambda,\lambda_\Lambda}$ is irreducible.

6.1. $G = \mathrm{SO}(2n,1)$.

Here $\mathfrak{g} = \mathfrak{so}(2n,1)$ is the set of matrices

$$\left(\begin{array}{c|c} 0 & {}^t z \\ \hline z & Y \end{array}\right)$$

with coefficients in $\mathbb{R}$, where z is a $(2n,1)$ column vector and Y is a $2n \times 2n$ matrix, such that $Y + {}^tY = 0$. The Lie algebras $\mathfrak{k}$ and $\mathfrak{p}$ correspond to the matrices above where respectively $z = 0$

and $Y=0$. The elements $a_{t_0}\in\mathfrak{a}\subset\mathfrak{p}$, with $t_0\in\mathbb{R}$, are the ones, when $z={}^t(t_0,0,\dots,0)$. The matrices in $\mathfrak{m}$ are of the form

$$\left(\begin{array}{cc|c} 0 & & 0 \\ & 0 & 0 \\ \hline 0 & 0 & X \end{array}\right)$$

where X is a $(2n-1)\times(2n-1)$ matrix, such that $X+{}^tX=0$. We choose the compact Cartan $\mathfrak{t}$ to be the matrix in $\mathfrak{k}$ such that $Y=\mathsf{diag}\left[\begin{pmatrix} & t_i \\ -t_i & \end{pmatrix}, t_i\in\mathbb{R},\ i=1,\dots,n\right]$. Finally $\mathfrak{h}:=(\mathfrak{t}\cap\mathfrak{m})\oplus\mathfrak{a}$. We define the fundamental weights as follows. For all $H\in\mathfrak{t}_{\mathbb{C}}$, $\varepsilon_j(H)=-it_j$, with $j=1,\dots,n$. For all $H\in\mathfrak{h}_{\mathbb{C}}$, $e_j(H)=-it_j$, with $j=2,\dots,n$, and $e_1(H)=t_0$. The corresponding roots vectors E_α in $\mathfrak{g}_{\mathbb{C}}$ are such that

$$\Big(E_{\pm\varepsilon_k}\Big)_{2k,1}=\Big(E_{\pm\varepsilon_k}\Big)_{1,2k}=1 \quad\text{and}\quad \Big(E_{\pm\varepsilon_k}\Big)_{2k+1,1}=\Big(E_{\pm\varepsilon_k}\Big)_{1,2k+1}=\mp i$$

and

$$\Big(E_{\varepsilon_k\pm\varepsilon_j}\Big)_{2k\to 2k+1,2j\to 2j+1}=-{}^t\Big(E_{\pm\varepsilon_k}\Big)_{2j\to 2j+1,2k\to 2k+1}=\begin{pmatrix}1 & \mp i\\ -i & \mp 1\end{pmatrix}.$$

Here for a matrix A we denote by $\Big(A\Big)_{k,j}$ the element in row k and column j, and $\Big(A\Big)_{k_1\to k_2,j_1\to j_2}$ the submatrix taking rows k_1 to k_2 and columns j_1 to j_2. Non defined elements are null. We have thus the following roots systems.

$$\Delta^+=\big\{\varepsilon_i\pm\varepsilon_j\mid i\neq j\big\}\cup\big\{\varepsilon_k\mid k\in[\![1,n]\!]\big\}.$$

The Cayley transform $\mathsf{Ad}u_{\varepsilon_1}$ send Δ^+ on

$$\Pi^+=\big\{e_i\pm e_j\mid i\neq j\big\}\cup\big\{e_k\mid k\in[\![1,n]\!]\big\},$$

such that e_1 is the real root. We denote by the same notation the fundamental weight and its restriction to other Cartan subalgebras. On $\mathfrak{h}_{\mathbb{C}}^-$, we have $e_i=\varepsilon_i$, for $i\in[\![1,n]\!]$.
As $\Lambda-\rho$ is Δ^+-dominant

$$\langle\mu_\Lambda,\varepsilon_n\rangle=\underbrace{\langle\Lambda-\rho,\varepsilon_n\rangle}_{\geq 0}+\underbrace{\langle 2\rho_n,\varepsilon_n\rangle}_{=1}\geq 1.$$

By the branching rules [BS79, Theorem 4.4], for all $\sigma\in\hat{M}$ with highest weight μ_σ,

$$\langle\mu_\sigma,\varepsilon_n\rangle\geq\langle\mu_\Lambda,\varepsilon_n\rangle\geq 1.$$

Fix one $\sigma\in\hat{M}(\tau)$. Recall that p_σ is an even function. The (positive) zeros of the polynomial part of the Plancherel density are (see for instance [Rob22, Section A.1])

$$(n-\frac{1}{2}-j+\langle\mu_\sigma,\varepsilon_{j+1}\rangle),\quad j=1,\dots,n-1.$$

This is a decreasing sequence of positive numbers, and we know that the last one is

$$(\frac{1}{2}+\langle\mu_\sigma,\varepsilon_n\rangle)\geq\frac{3}{2}$$

The second factor of $p_\sigma(\lambda\alpha)$ for $\lambda\in\mathbb{C}$ is either $\tanh\pi\lambda$ or $\coth\pi\lambda$. Thus either $\frac{1}{2}$ or 1 is a pole of p_σ. We choose λ_Λ to be this pole. The infinitesimal character of $\mathrm{Ind}_{MAN}^G(\sigma\otimes\lambda_\Lambda\otimes 1)$ is

$$\chi_{\sigma,\lambda_\Lambda}=\lambda e_1+\mu_\sigma+\rho_{\mathfrak{m}}$$

We know that

$$\langle \chi_{\sigma,\lambda_\Lambda}, e_1\rangle = \lambda < \frac{3}{2} \leq \langle \chi_{\sigma,\lambda_\Lambda}, e_n\rangle = (\frac{1}{2} + \langle \mu_\sigma, \varepsilon_n\rangle) \ .$$

Following [Col85, Theorem 4.3.1] (recalled in [Rob23, Theorem 2]), $\chi_{\sigma,\lambda_\Lambda}$ corresponds to $\gamma_{0,n}$ (notation in [Rob23, Theorem 3]). So the composition series is

$$\pi_{0,n} = \begin{array}{|c|c|c|} \hline \multicolumn{3}{|c|}{\overline{\pi}_{0,n}} \\ \hline \overline{\pi}_0 & \oplus & \overline{\pi}_1 \\ \hline \end{array}$$

As in [Rob22, Section 4], $\mathscr{E}^{\sigma,\lambda_\Lambda}_{\tau_\Lambda}$ is the irreducible subquotient where τ_Λ is. To see where it is, we check an other principal series representation. Namely let

$$\mu_{\sigma_\rightarrow} := \mu_\sigma + (\lambda_\Lambda - \langle \mu_\sigma, \varepsilon_n\rangle)\varepsilon_n$$
$$\lambda_\rightarrow := \langle \mu_\sigma, \varepsilon_n\rangle \ .$$

Then the principal series representation $\mathrm{Ind}^G_{MAN}(\sigma_\rightarrow \otimes \lambda_\rightarrow \otimes 1)$ has infinitesimal character

$$\chi_\rightarrow = \langle \mu_\sigma, \varepsilon_n\rangle e_1 + \mu_\sigma + \lambda e_n \ .$$

We shifted the coefficient of e_1. We have

$$\langle \chi_\rightarrow, e_n\rangle = \lambda_\Lambda < \frac{3}{2} \leq \langle \chi_\rightarrow, e_1\rangle = \langle \mu_\sigma + \rho_{\mathfrak{m}}, e_n\rangle < \langle \mu_\sigma + \rho_{\mathfrak{m}}, \varepsilon_{n-1}\rangle = \langle \chi_\rightarrow, e_{n-1}\rangle$$

So $\chi_\rightarrow$ correspond to $\gamma_{0,n-1}$ and has the same composition series as

$$\pi_{0,n-1} = \begin{array}{|c|} \hline \overline{\pi}_{0,n-1} \\ \hline \| \ \overline{\pi}_{0,n} \\ \hline \end{array}$$

We also know that $\sigma_\rightarrow \notin \hat{M}(\tau_\Lambda)$, because $\langle \sigma_\rightarrow, \varepsilon_n\rangle = \lambda_\Lambda < \langle \mu_\Lambda, \varepsilon_n\rangle$. Thanks to Frobenius Theorem,

$$\tau_\Lambda \notin \mathrm{Ind}^G_{MAN}(\sigma_\rightarrow \otimes \lambda_\rightarrow \otimes 1)$$

So τ_Λ is not in the irreducible subquotient corresponding to $\overline{\pi}_{0,n}$. There are only left the irreducible subquotients corresponding to $\overline{\pi}_0$ or $\overline{\pi}_1$. They are both discrete series representations. The Harish-Chandra parameter is Λ because the Blattner parameter is μ_λ. This proves the following theorem.

Theorem 3
The residue representation $\mathscr{E}^{\sigma,\lambda_\Lambda}_{\tau_\Lambda}$ is unitarly equivalent with the discrete series representation of Harish-Chandra parameter Λ.

One can remark that for this case, σ can be choose arbitrary in $\hat{M}(\tau)$ and $\lambda_\Lambda = \frac{1}{2}$ or 1.

6.2. $G = \mathrm{SU}(n,1)$**.** Let $\mathfrak{su}(n)$ be the space of $n \times n$ matrices Y such that $Y + {}^t\overline{Y} = 0$. Here $\mathfrak{g} = \mathfrak{su}(1,n)$ is the set of matrices

$$\left(\begin{array}{c|c} w & {}^t\overline{z} \\ \hline z & Y \end{array} \right)$$

with coefficients in $\mathbb{C}$, where z is a $(n,1)$ column vector, $w \in i\mathbb{R}$ and $Y \in \mathfrak{su}(n)$. The Lie algebra $\mathfrak{k}$ (respectively $\mathfrak{p}$) corresponds to the matrices above where $z = 0$ (respectively $Y = 0$

and $w = 0$). The elements $a_{t_0} \in \mathfrak{a} \subset \mathfrak{p}$, with $t_0 \in \mathbb{C}$, are the ones, when $z = {}^t(0, \dots, 0, t_0)$. We have

$$\mathfrak{m} := \left(\begin{array}{c|c|c} w & & \\ \hline & \mathfrak{su}(n-1) & \\ \hline & & w \end{array} \right) .$$

We choose the compact Cartan $\mathfrak{t}$ to be the matrix in $\mathfrak{k}$ such that $Y = \mathsf{diag}\,[t_i,\ t_i \in i\mathbb{R},\ i = 1, \dots, n+1]$ and the noncompact Cartan $\mathfrak{h} := (\mathfrak{t} \cap \mathfrak{m}) \oplus \mathfrak{a}$. We define the fundamental weights as follows. For all $H \in \mathfrak{t}_{\mathbb{C}}$, $\varepsilon_j(H) = t_j$, with $j = 1, \dots, n+1$. For all $H \in \mathfrak{h}_{\mathbb{C}}$, $e_j(H) = t_j$, with $j = 1, \dots, n$ and $e_{n+1}(H) = t_0$. The corresponding roots vectors $E_{\varepsilon_k - \varepsilon_j}$ verify

$$\left(E_{\varepsilon_k - \varepsilon_j} \right)_{k,j} = 1 .$$

We have thus the following roots systems.

$$\Delta^+ = \left\{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1 \right\} .$$

The Cayley transform $\mathsf{Ad} u_{\varepsilon_1 - \varepsilon_{n+1}}$ send Δ^+ on

$$\Pi^+ = \left\{ e_i - e_j \mid 1 \leq i < j \leq n+1 \right\} ,$$

such that $e_1 - e_{n+1}$ is the real root (corresponding to the longest restricted root α). We denote by the same notation the fundamental weight and its restriction to other Cartan subalgebras. On $\mathfrak{h}_{\mathbb{C}}^-$, we have $e_i = \varepsilon_i$, for $i \in [\![2, n]\!]$.

Trace of matrices in $\mathfrak{h}_{\mathbb{C}}^*$ are 0, so $\sum\limits_{j=1}^{n+1} \varepsilon_j$. Thus, without loss of generality, we can suppose that $\langle \mu_\Lambda, \varepsilon_{n+1} \rangle = 0$.

As $\Lambda - \rho$ is Δ^+ dominant, for $1 \leq i < j < n+1$,

$$\langle \mu_\Lambda, \varepsilon_i - \varepsilon_j \rangle = \underbrace{\langle \Lambda - \rho, \varepsilon_i - \varepsilon_j \rangle}_{\geq 0} + \underbrace{\langle 2\rho_n, \varepsilon_i - \varepsilon_j \rangle}_{=0} \geq 0 \tag{21}$$

$$\langle \mu_\Lambda, \varepsilon_i \rangle \geq \langle \mu_\Lambda, \varepsilon_n \rangle = \langle \mu_\Lambda, \varepsilon_n - \varepsilon_{n+1} \rangle = \underbrace{\langle \Lambda - \rho, \varepsilon_n - \varepsilon_{n+1} \rangle}_{\geq 0} + n + 1 \geq n+1 , \tag{22}$$

because $2\rho_n = \sum\limits_{i=1}^{n} \varepsilon_i - n\varepsilon_{n+1}$. Let $\sigma_\Lambda \in \hat{M}(\tau)$ such that,

$$\mu_{\sigma_\Lambda} = \sum_{i=2}^{n} \langle \mu_\Lambda, \varepsilon_{i-1} \rangle \varepsilon_i + \frac{\langle \mu_\Lambda, \varepsilon_n \rangle}{2} (\varepsilon_1 + \varepsilon_{n-1}) .$$

The zeros of the polynomial part of the Plancherel density p_{σ_Λ} are the numbers

$$\frac{n}{2} - j - \frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n \rangle}{2} + \langle \mu_{\sigma_\Lambda}, \varepsilon_{j+1} \rangle \quad , \quad j = 1, \dots, n-1 . \tag{23}$$

This is a decreasing sequence of positive numbers. See [Rob22, Section A.2] Because of (22), the last number is

$$\frac{n}{2} - n + 1 - \frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n \rangle}{2} + \langle \mu_{\sigma_\Lambda}, \varepsilon_n \rangle = -\frac{n}{2} + 1 + \frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n \rangle}{2} \geq -\frac{n}{2} + 1 + \frac{n+1}{2} \geq \frac{3}{2} .$$

The second factor of $p_{\sigma_\Lambda}(\lambda\alpha)$ for $\lambda \in \mathbb{C}$ is either $\tanh \pi\lambda$ or $\coth \pi\lambda$. Thus either $\frac{1}{2}$ or 1 is a pole of p_σ. We choose λ_Λ to be this pole. The infinitesimal character of $\mathrm{Ind}_{MAN}^{G}(\sigma_\Lambda \otimes \lambda_\Lambda \otimes 1)$ is

$$\chi_{\sigma_\Lambda,\lambda_\Lambda} = \lambda_\Lambda(e_1 - e_{n+1}) + \mu_\sigma + \rho_{\mathfrak{m}}$$

We have

$$\langle \chi_{\sigma_\Lambda,\lambda_\Lambda}, e_1\rangle = \lambda + \frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n\rangle}{2} \Bigg< \underbrace{\frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n\rangle - n - 1}{2}}_{\geq 0} + \underbrace{\frac{3}{2}}_{\lambda_\Lambda} = -\frac{n}{2} + \langle \mu_{\sigma_\Lambda}, \varepsilon_n\rangle + 1 = \langle \chi_{\sigma_\Lambda,\lambda_\Lambda}, e_n\rangle$$

and

$$\langle \chi_{\sigma_\Lambda,\lambda_\Lambda}, e_1\rangle \Bigg> -\lambda + \frac{\langle \mu_{\sigma_\Lambda}, \varepsilon_n\rangle}{2} = \langle \chi_{\sigma_\Lambda,\lambda_\Lambda}, e_{n+1}\rangle$$

So following [Col85, Theorem 4.3.1] (recalled in [Rob23, Theorem 2]), $\chi_{\sigma,\lambda_\Lambda}$ corresponds to $\gamma_{n-1,1}$ (notation in [Rob23, Theorem 4]). So the composition series is

$$\pi_{n-1,1} = \begin{array}{|c|c|c|}\hline \multicolumn{3}{|c|}{\overline{\pi}_{n-1,1}} \\ \hline \overline{\pi}_{n-1} & \oplus & \overline{\pi}_n \\ \hline \end{array}$$

As in [Rob22, Section 4], $\mathscr{E}_{\tau_\Lambda}^{\sigma_\Lambda,\lambda_\Lambda}$ is the irreducible subquotient where τ_Λ is. To see where it is, we check an other principal series representation. Namely let

$$\begin{aligned}\mu_{\sigma_\to} &:= \mu_{\sigma_\Lambda} + \left(\lambda_\Lambda - 1 + \frac{\langle \mu_\Lambda, \varepsilon_n\rangle + n}{2} - \langle \mu_\Lambda, \varepsilon_{n-1}\rangle\right)\varepsilon_n \\ &\quad + \frac{-2\lambda_\Lambda + 2\langle \mu_\Lambda, \varepsilon_{n-1}\rangle - n + 2 - \langle \mu_\Lambda, \varepsilon_n\rangle}{4}(\varepsilon_1 + \varepsilon_{n+1}) \\ \lambda_\to &:= \frac{-2\lambda_\Lambda - 2\langle \mu_\Lambda, \varepsilon_{n-1}\rangle + n - 2 + \langle \mu_\Lambda, \varepsilon_n\rangle}{4} \ .\end{aligned}$$

Then the principal series representation $\mathrm{Ind}_{MAN}^{G}(\sigma_\to \otimes \lambda_\to \otimes 1)$ has infinitesimal character

$$\chi_\to = \left(-\lambda_\Lambda + \frac{\langle \mu_\Lambda, \varepsilon_n\rangle}{2}\right) e_1 + \left(\langle \mu_\Lambda, \varepsilon_{n-1}\rangle - \frac{n}{2} + 1\right) e_{n+1} + \sum_{i=2}^{n-1} \langle \mu_{\sigma_\Lambda} + \rho_{\mathfrak{m}}, \varepsilon_i\rangle\, \varepsilon_i + \left(\lambda_\Lambda + \frac{\langle \mu_\Lambda, \varepsilon_n\rangle}{2}\right)\varepsilon_n \ .$$

We shifted the coefficient of ε_1. We have

$$\langle \chi_\to, e_1\rangle < \frac{3}{2} \leq \langle \chi_\to, e_n\rangle < \langle \chi_\to, e_{n+1}\rangle \ .$$

So $\chi_\to$ correspond to $\gamma_{n-2,1}$ and has the same composition series as

$$\pi_{n-2,1} = \begin{array}{|c|c|c|}\hline \multicolumn{3}{|c|}{\overline{\pi}_{n-2,1}} \\ \hline \overline{\pi}_{n-1,1} & \oplus & \overline{\pi}_{n-2,2} \\ \hline \multicolumn{3}{|c|}{\overline{\pi}_{n-1}} \\ \hline \end{array}$$

But $\langle \mu_{\sigma_\to}, \varepsilon_n\rangle = \lambda_\Lambda - 1 + \dfrac{\langle \mu_\Lambda, \varepsilon_n\rangle + n}{2} = \overbrace{\lambda_\Lambda - \frac{3}{2}}^{<0} + \dfrac{\langle \mu_\Lambda, \varepsilon_n\rangle + \overbrace{n+1}^{\leq \langle \mu_\Lambda, \varepsilon_n\rangle}}{2} \Bigg< \langle \mu_\Lambda, \varepsilon_n\rangle$. So because of the branching rules in [BS79, Theorem 4.4], $\sigma_\to \notin \hat{M}(\tau_\Lambda)$. Thanks to Frobenius reciprocity

Theorem,

$$\tau_\Lambda \notin \mathrm{Ind}_{MAN}^{G}(\sigma_\to \otimes \lambda_\to \otimes 1)$$

So τ_Λ is not in the irreducible subquotient corresponding to the irreducibles in $\pi_{n-2,1}$. There is only left the irreducible subquotient corresponding to $\overline{\pi}_n$. The Harish-Chandra parameter is Λ because the Blattner parameter is μ_λ. This proves the following theorem.

Theorem 4
With the notations above, the residue representation $\mathscr{E}_{\tau_\Lambda}^{\sigma_\Lambda,\lambda_\Lambda}$ is unitarily equivalent with the discrete series representation of Harish-Chandra parameter Λ.

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