Research statement

Simon ROBY

Résumé en Français

Mon domaine de recherche est l'analyse harmonique sur les espaces symétriques et la théorie des représentations des groupes de Lie, suivant les travaux pionniers de Harish-Chandra, Godement et Helgason.

Ma thèse [59] portait sur les résonances du Laplacien sur les fibrés vectoriels homogènes. Le but étaient de trouver une continuation méromorphe de la resolvante du Laplacien, agissant sur les sections d'un fibré vectoriel homogène au dessus des espaces Riemanniens symétriques de rang un. Les pôles de cette fonction méromorphe sont appelés les résonances. De chaque résonance est issue de manière canonique une représentation. J'ai determiné les paramètres de Langlands et le front d'onde de ces représentations. L'article [60] issu de ma thèse donne l'ensemble des résonances et étudie les représentations avec un condition supplémentaire sur le fibré vectoriel. L'article [61] traite du cas général. Dans [63] à paraître, on peut voir que toutes les représentations issues de la série discrète, des groupes de Lie de rang un, peuvent être réalisées de cette manière. Cela montre que l'on peut porter un intérêt particulier à ces représentations. Dans [62], j'utilise la même méthode pour trouver les résonances et les représentations issues des résidus sur l'espace Anti-de Sitter. C'était le premier cas d'espace non Riemannien. Dans le futur, j'aimerai essayer de comprendre les liens entre les représentations des résonances dites *quantiques*, étudiées dans ma thèse, et les états des résonances dites *classiques*, aussi appelées résonances de Ruelle, pôles de la continuation méromorphe d'un flot sur les espaces localement symétriques. Une correspondance par les représentations issues de chaque côté semble possible. Les cas des espaces Riemanniens de rang réel supérieur à 3 ainsi que des autres espaces symmétriques non-Riemannien restent toujours ouverts. Une idée proposée dans [6] indique que l'on peut aussi utiliser la correspondance de Howe pour obtenir une notion de résonances pour les opérateurs de Capelli. Cela pourrait généraliser le calcul des résonances à d'autres opérateurs.

Introduction to my research

Background. My research field is harmonic analysis on symmetric spaces and representation theory of Lie groups, following the pioneering works of Harish-Chandra, Godement and Helgason.

I consider homogeneous vector bundles on symmetric spaces. Specifically, I am interested in the action of the positive Laplace operator Δ on the smooth sections. In my thesis, I studied the so-called rank-one case. I computed the meromorphic extension of the resolvent of Δ using the works of Camporesi (see [7, 9]), who generalised some fundamental results of Helgason to homogeneous vector bundles, and the subrepresentation theorem for principal series representations (see for instance [42]). The poles of that extension are called *resonances* (or quantum resonances). They exist because the Plancherel measure has singularities. To each resonance one can associate a residue operator whose range is a representation space of the group G of isometries of the symmetric space. One wants to find out which of them are finite or infinite dimensional, irreducible, or unitary. If irreducible then one wants to identify them in terms of their Langlands parameters and compute their wave front sets.

Identifying the image of the residue operators gives important additional information on the image of the vector-valued Poisson transform, whose study was initiated in [54].

Resonances. The notion of resonances was introduced in quantum mechanics to study metastable states of a system, for instance system of particles, that is long-lived states from which the system deviates only with sufficiently strong disturbances. Some mathematical aspects of the theory of resonances can already be traced to the work of Weisskopf and Wigner [68] in 1930. Its modern mathematical form was probably first laid down by Simon [64] in the late 1970s. Mathematically, resonances are spectral objects attached to linear operators on non-compact domains and appear as poles of the meromorphic continuation of the resolvent of these operators. What in mathematics one calls resonance is the energy of a resonant state in physics. So, resonances replace discrete eigenvalues of linear operators.

The study of resonances evolved from an investigation of the Schrödinger operators on the Euclidean spaces to a study of the Laplacian on a curved space, like a hyperbolic or asymptotically hyperbolic manifold, symmetric, locally symmetric space, or Damek-Ricci spaces. The results play an important role in several physical contexts (propagation and evolution of states having certain properties of regularity to infinity, formulas of traces, quantum dynamics).

In a typical setting, one works on a complete Riemannian manifold X with a finite geometry, for which the positive Laplacian Δ is an essentially self-adjoint operator on the Hilbert space $L^2(X)$ of square integrable functions on X. We suppose that Δ has a continuous spectrum $[\rho_X, +\infty)$, with $\rho_X \geq 0$. The resolvent $R(z) = (\Delta_X - \rho_X - z^2)^{-1}$ of the shifted Laplacian (or Helmholtz operator) $\Delta - \rho_X$ is then a holomorphic function of z on the upper (and on the lower) complex half plane, with values in the space of bounded linear operators on $L^2(X)$. Let the resolvent act, not on the entire $L^2(X)$, but on a dense subspace of $L^2(X)$, for instance the space $C_c^\infty(X)$ of compactly supported smooth functions on X or on some suitable weighted L^2 space. Then the map $z \mapsto R(z)$ might admit a meromorphic extension across $\mathbb{R}$ to a larger domain in $\mathbb{C}$ or to a cover of such a domain. The poles, if they exist, are the resonances (also called quantum resonances or scattering poles) of Δ.[1]

The basic questions concern the existence of the meromorphic extension of the resolvent, the distribution and counting properties of the resonances, the rank and interpretation of the residue operators associated with the resonances.

Resonances are linked to interesting geometric, dynamical and analytic objects. For instance, on hyperbolic manifolds, they encode important information on the dynamics of the space and reflect the geometric properties of the geodesics. The works of Patterson-Perry [55]

[1] For simplicity, in the following we will always assume to have shifted the Laplacian so that its spectrum has bottom at 0, and to have changed variables $z \to z^2$ for the resolvent so that, as above, the resolvent is analytic way from the real axis.

and Borthwick-Judge-Perry [4] have shown that the resonances of the Laplacian are connected to the zero set of Selberg's zeta function, which relates the length spectrum of the geodesics and spectral data. Counting properties of the resonances play an important role in several applications of the theory of dynamical systems and in number theory. All these objects are nowadays intensively studied, using different techniques and from different viewpoints, by many authors. Among them, we should mention Borthwick, Datchev, Dyatlov, Faure, Guillarmou, Hilgert, Mazzeo, Müller, Nonnenmacher, Olbrich, Perry, Vasy, Weich, Zworski. Nevertheless, most of the studies and of the results are related to low rank and low dimensional situations. Very little is known for locally symmetric spaces of higher rank. In fact, even the model situation of symmetric spaces is far from being understood either when the space has rank bigger than 1 or when the Laplacian acts on vector valued objects, like vector fields or forms, rather than on functions. Model spaces, like Riemannian symmetric spaces of the non-compact type, are important to understand situations on more complicate geometries.

A Riemannian symmetric space of the non-compact type is a homogeneous space of the form $X = G/K$ where G is a connected noncompact real semisimple Lie group with finite center and K is a maximal compact subgroup of G. The maximal flat subspaces of X (the Cartan subspaces) have constant dimension, called the (real) rank of the space. When this rank is one, X is real, complex, quaternionic hyperbolic spaces or the octonionic plan. The resonances and the residue representations in this case were determined in [33].

For general Riemannian symmetric spaces of the noncompact type, the study of the analytic extension of the resolvent of the Laplacian was independently started by Mazzeo and Vasy, [48], and Strohmaier, [65]. The analysis carried out in [48] combines microlocal techniques and an adaptation of the scattering method of complex scaling of Aguilar-Balslev-Combes, see e.g. [37]. In [65], Strohmaier used the Helgason-Fourier analysis, see e.g. [30, Chapter 3]. Both [48] and [65] proved that the resolvent $z \mapsto R(z)$ of the shifted Laplacian $\Delta_X - \rho_X$ admits a holomorphic extension across $\mathbb{R}$. The domain of the extension depends on the parity of the real rank of the symmetric space X. This parity plays for the analytic extension of the Laplacian on symmetric spaces the same role as the dimension of $\mathbb{R}^n$ in the case of the Euclidean Laplacian on (see e.g. [51, §1,6]). Both articles also proved that the resonances of the extended resolvent, if any, must be located on a certain half-line, see also [33, Corollary 2.2. and Remark 2.3]. There might be no resonances at all. This happens for instance, when the symmetric spaces have even multiplicities (e.g. if G has a complex structure or for the hyperbolic spaces $\mathbb{H}^n$ with n even, as it was proved earlier by [24]). If this can be interesting, symmetric spaces of even multiplicities are very easy to deal with and definitely not typical. In fact, the articles [48] and [65] could not answer the basic question if for a general symmetric space of rank higher than 1 there are any resonances.

My first goal was to try to extend these results to homogeneous vector bundles over X (always in rank-one). These bundles are determined by a representation τ of K. In this context, I used the vector valued spherical functions. They are central to my work because they connect the resonances and the representations. For the rank-one case, the resonances are listed in [60]. It also study and identify completely the representations Π arising from the resonances in the case when τ contain the trivial representation of M. The representation theoretic interpretation of these operators is connected with the analysis of the composition series of the principal series representations. In general, this is complicated and involves a case-by-case analysis (see [14]). But the structure can be unified in our case. [60] computes the wave front set of the residue representations. I wanted to show how to identify the residue representations in some examples with no intersection the general case study in my thesis. Finally in [61], I found a algorithm, which aims at determining the residue representations (Langlands parameter, wave front sets) for any representation of K. I compute them concretely for the p-forms representations (see [56]) for the classical real-ranked-one Lie groups.

Projects

In the following I briefly describe some projects I am working on, beginning with those which are closed to completion. This list is by no means rigid or complete in the sense that I am very open to new ideas and direction.

Symmetric spaces of rank 2. I would like to complete the study of the resonance representations initiated by Joachim Hilgert, Angela Pasquale and Tomasz Przebinda in [36], [35] and [34]. Here the main case is the exceptional group G_2. As in the previous cases I would like to identify the Langlands parameters of the spherical resonance representations, use Vogan's classification [66] to see which of them are unitarizable and compute their wave front sets. There are some hidden symmetries here too. The group G_2 is contained in $\mathrm{O}_{4,3}$ which is a member of the dual pair $(\mathrm{O}_{4,3}, \mathrm{Sp}_2(\mathbb{R}))$. This inclusion was exploited in [58] and I would like to continue that inquiry. I am currently working on a generalisation of the computations of the resonances. The computations for G_2 seem hard but doable.

Quantum Classical correspondances. In my thesis, I studied the "quantum" resonances of the Laplacian acting on section on some homogeneous vector bundles. These quantum resonances are roughly the generalisation of the eigenvalues of the Laplacian on compact manifold. Resonances are linked to interesting geometric, dynamical and analytic objects. For instance, on hyperbolic manifolds, they encode important information on the dynamics of the space and reflect the geometric properties of the geodesics. The works of Patterson-Perry [55] and Borthwick-Judge-Perry [4] have shown that the resonances of the Laplacian are connected to the zero set of Selberg's zeta function, which relates the length spectrum of the geodesics and spectral data. Counting properties of the resonances play an important role in several applications of the theory of dynamical systems and in number theory. All these objects are nowadays intensively studied, using different techniques and from different viewpoints, by many authors. Among them, we should mention [5, 16, 20, 23, 24, 49]. Nevertheless, most of the studies and of the results are related to low rank and low dimensional situations. Very little is known for locally symmetric spaces of higher rank. In fact, even the model situation of symmetric spaces is far from being understood either when the space has rank bigger than 1 or when the Laplacian acts on vector valued objects, like vector fields or forms, rather than on functions. Professor Tobias Weich has a more general project on higher dimensional locally symmetric spaces which is very interesting for me. His knowledge and his experience would be very appreciated in my future career. Model spaces, like Riemannian symmetric spaces of the non-compact type (what I studied), are important to understand situations on more complicate geometries, like locally symmetric spaces.

Let's explain the more concrete project, which can be a path to link my work with the research carried in Paderborn Universität. We consider a locally Riemannian symmetric space $\mathcal{M}$ of rank 1. Then $\mathcal{M} = \Gamma\backslash G/K$, where G is semisimple Lie group of finite center, K a maximal compact subgroup of G and Γ a torsion free cocompact discrete subgroup of G. It's possible to consider a more general (or different) setting, for instance $\mathcal{M}$ a Riemannian compact manifold without boundary. But for this project, we want to stay in this rank one case. Thanks to the Riemannian metric on $\mathcal{M}$, one can define the Laplace-Beltrami operator $\Delta : \mathrm{C}^\infty(\mathcal{M}) \to \mathrm{C}^\infty(\mathcal{M})$ and the geodesic flow φ_t on the unit cosphere bundle $\mathbf{S}^*\mathcal{M}$ generated by a smooth vector field $X : \mathrm{C}^\infty(\mathbf{S}^*\mathcal{M}) \to \mathrm{C}^\infty(\mathbf{S}^*\mathcal{M})$. If we suppose that $\mathcal{M}$ is compact, the spectrum of Δ is purely discrete and of finite multiplicity. The *quantum-classical correspondence* consists of relations between the discrete spectrum of Δ and the properties of the geodesic flow. Thanks to the semisimple structure of G, we have the Iwasawa decomposition

$$G = KAN^+ = KAN^-,$$

where A is a Lie group of exponential type and of dimension 1 (= real rank of G) as a manifold, an $N^\pm$ is a nilpotent Lie subgroup with Lie algebras

$$\mathfrak{n}^\pm := \sum_{\alpha \in \mathcal{S}^\pm} \mathfrak{g}_\alpha.$$

Here $S^\pm$ represents the set of positive/negative roots and $\mathfrak{g}_\alpha$ is the space of root vectors associated to the root α. Set $M = \mathrm{Z}_K(\mathfrak{a})$. We have $\mathbf{S}^*\mathcal{M} = \Gamma\backslash G/M$, and, thanks to the Bruhat decomposition,

$$\mathfrak{g}/\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- \ .$$

So under our assumptions and taking the flow φ_t associated to the A action is Anosov, i.e.

$$\mathrm{T}(\mathbf{S}^*\mathcal{M}) = \mathrm{E}_0 \oplus \mathrm{E}_+ \oplus \mathrm{E}_- \ ,$$

where $\mathrm{E}_0 := G \times_{\mathsf{Ad}(M)} \mathfrak{a}$ is the neutral bundle of the Anosov flow and $\mathrm{E}_\pm := G \times_{\mathsf{Ad}(M)} \mathfrak{n}^\pm$ are the stable and the unstable bundles of the Anosov flow. We have also the dual splitting:

$$\mathrm{T}^*(\mathbf{S}^*\mathcal{M}) = \mathrm{E}_0^* \oplus \mathrm{E}_+^* \oplus \mathrm{E}_-^* \ ,$$

where $\mathrm{E}_i^*(\mathrm{E}_j) = 0$ if $i \neq j$, with $i, j \in \{+, -, 0\}$.

In the case of functions, also called *scalar case*, one defines a Policott-Ruelle resonance as a complex number $\lambda \in \mathbb{C}$ such that the space

$$\mathsf{Res}_X(\lambda) := \left\{ u \in \mathcal{D}'(\mathbf{S}^*\mathcal{M}) \mid (X + \lambda)u = 0, \mathsf{WF}(u) \subset E_-^* \right\}$$

is non trivial. Here $D'(\mathbf{S}^*\mathcal{M})$ is for the space of distributions on $\mathbf{S}^*\mathcal{M}$ and $\mathsf{WF}(u)$ its Hörmander wave front set (See [38, Chater 8] for the definition). Without going deeper, one can prove (see for instance [16, 2A]) that this space can be described by the spaces of *first band resonant states*:

$$\mathsf{Res}_X^0(\lambda) := \left\{ u \in \mathcal{D}'(\mathbf{S}^*\mathcal{M}) \mid (X + \lambda)u = 0, \mathsf{WF}(u) \subset E_-^*, \text{ and } \forall \mathbf{x} \in \Gamma^\infty(E_-),\ \mathbf{x}u = 0 \right\}$$

A complex number $\lambda \in \mathbb{C}$ such that $\mathsf{Res}_X^0(\lambda) \neq \{0\}$ is called a *first band resonance*. These are the closest Polycott-Ruelle resonances to the imaginary axes.

In [16], for $\mathrm{PSO}(u,1)$, the authors prove the fundamental results: if $\lambda \in \mathbb{C}$ is a first band Pollicott–Ruelle resonance, then the pushforward $\pi_* : D'(\mathbf{S}^*\mathcal{M}) \to D'(\mathcal{M})$ of the canonical bundle projection $\pi : \mathbf{S}^*\mathcal{M} \to \mathcal{M}$ restricts to

$$\pi_* : \mathsf{Res}^0(\lambda) \to \ker_{L^2(M)}(\Delta - \mu(\lambda)) ,$$

where $\mu(\lambda) = -(\lambda+\rho)^2 + \rho^2$ and $\rho = \frac{\dim(\mathcal{M})-1}{2}$. Moreover, excepting some exceptional values ($\lambda \notin -\rho - \frac{1}{2}\mathbb{N}_0$), π_* is an isomorphism. In the scalar case, lots of papers followed this one, in difference settings and with different degrees of generalities. Among the works concerned, one can cite [22, 21, 26, 25] and a complete recent review [32]. The correspondences established between the spectral values of the Laplacian Δ and the properties of the geodesic flow ϕ_t are called in the literature *quantum-classical correspondence*. The understanding of the space of resonant states is a crucial step.

In [43], the authors extend these results to the setting of associated vector bundles over arbitrary compact Riemannian locally symmetric spaces of rank 1. We will refer to this situation as the *bundle case*. Without going into the details, they succeed to have a map similar to π_* above. One important tool is the vector valued Poisson transform introduced by Olbrich in his PhD thesis [54]. The results, as for the papers cited in the scalar case, are provided except for a number of some exceptional spectral values. These values correspond to the ones, for which the Poisson transform is not an isomorphism.

One solution for these exceptional spectral values has been given in [2]. They used the structure of the spherical principal series and the vector valued Poisson transform. They found interesting results about the resonance states, namely they described them as discrete series of certain pseudo-Riemannian symmetric spaces.

The vector bundle case seems much more difficult. In fact, similar method as [2] has been used for selected residue representations, using the structure of spherical principal series, in [60]. The results was pretty general and not too technical too explain. But for general residue representation, the result seem to be impossible to give in a generic way. An algorithm to find the residue representations in any case has been developed in [61]. The paper also treated some special case.

So using the Poisson transform as in [2] may give similar results for the exceptional parameters using the theory developed in [43] for some specific vector bundles. This project is technically very challenging and will require a lot of explicit knowledge about representation theory, that I have acquired in my previous research projects. But it will help to understand why should be interesting to have a precise understanding of these exceptional spectral parameters in this setting. To succeed it is important to understand the setting developed by Küster and Weich in [43] and to know the exact composition series of the principal series representations and the vector valued Poisson transforms used to match with the resonant states. The second point is the one I bring my knowledge into the subject.

A connection with Howe's correspondence. The isometry groups $\mathrm{O}_{p,1}$, $\mathrm{U}_{1,1}$, $\mathrm{Sp}_{p,1}$ of the classical symmetric spaces are members of the following reductive dual pairs

$$(G, G') = (\mathrm{O}_{p,1}, \mathrm{Sp}_2(\mathbb{R})) , \quad (\mathrm{U}_{p,1}, \mathrm{U}_{1,1}) , \quad (\mathrm{Sp}_{p,1}, \mathrm{O}_4^*) .$$

Moreover, as I am going to describe below, the residue representation Π of G I obtained, extend to a representation of $G \times G'$ of the form $\Pi \otimes \Pi'$ for the representation Π' of G' which occurs in Howe duality. Thus there is a "hidden symmetry" which I would like to bring to light.

Let us take a look for example at the first dual pair $(G, G') = (\mathrm{O}_{p,1}, \mathrm{Sp}_2(\mathbb{R}))$. The group G acts on $\mathbb{R}^p \oplus \mathbb{R}$ preserving the quadratic form $r_{p,1}^2(x, y) = x \cdot x - y^2$ and therefore also commuting with the Dalambertian $\Box = \Delta_{p,1} = \partial_{x_1}^2 + ... + \partial_{x_p}^2 - \partial_y^2$. These two operators generate a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, which turns out to be $\mathfrak{g}'$, the Lie algebra of G'. It should not be too difficult to identify Π' and describe the resulting $G \times G'$-module in general.

However, since the group G' is smaller than G, it also is tempting to reverse the procedure, work with the smaller group G', and then use Howe's correspondence to obtain resonances for the larger group G, no matter what is the rank of the corresponding symmetric space. The first example would be $(\mathrm{O}_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$, with $2 \leq q \leq p$.

Similarly the exceptional group F_4 fits into the theory of exceptional dual pairs developed by Benedict Gross, Gordan Savin and others.

An interesting way to expand the results of my thesis on other operators has been shown in [6]. The authors succeed to determine the resonance of Capelli operators in a small case.

One great benefit from the Spectral Theorem is the functional calculus. Given a bounded operator A on a Hilbert space V one defines $f(A)$ for a holomorphic function f in a neighborhood of the spectrum of A. If the spectrum consists of eigenvalues, then we decompose the space into direct sum of eigenspaces,

$$V = \bigoplus_{\alpha} V_\alpha$$

and set

$$f(A) = \bigoplus_{\alpha} f(\alpha) I_{V_\alpha}.$$

If B is another bounded operator commuting with A then we use the joint decomposition

$$V = \bigoplus_{\alpha,\beta} V_{\alpha,\beta}$$

where

$$A|_{V_{\alpha,\beta}} = \alpha I_{V_{\alpha,\beta}} \quad , \quad B|_{V_{\alpha,\beta}} = \beta I_{V_{\alpha,\beta}}.$$

Since the resonances are supposed to play the role of the non-existing eigenvalues and the resonance spaces are supposed to replace the eigenspaces, it seems reasonable to ask for an iterated resonance of two commuting operators A and B. More precisely, compute its resonances and corresponding resonance spaces for A. Each such space should be preserved by B. Then one should compute the resonance of B. Reversing B and A should give the same spaces. One good place to start experimentation is Howe's theory of dual pairs. Let ω be the Weil representation of the metaplecitc group $\widetilde{\mathrm{Sp}}(W)$. Let (G, G') be a reductive dual pair in $\mathrm{Sp}(W)$. Then we have a commutative algebra

$$\omega(\mathcal{U}(\mathfrak{g})^G) = \omega(\mathcal{U}(\mathfrak{g}')^{G'})$$

of Capelli operators. It is generated by n elements, where n is the minimum of the rank of $\mathfrak{g}$ and $\mathfrak{g}'$. One could pick n generators and play with them. [6] deals with $n = 1$. The case $n = 2$ is wide open. For the dual pair $\big(\mathrm{Sp}_2(\mathbb{R}), \mathrm{O}(2, 2)\big)$ all representations in Howe's correspondence are described. One may try that.

A connection with super Lie algebras. As is known from the works of Olbrich [54], the algebra of operators on a homogeneous vector bundle over a symmetric space that commute with the group action may be quite large and hard to handle. I would like to study examples of algebra of differential operators having some nice structure. I'll explain it on an example following [39, sec. 7].

The Lorentz group $O_{3,1}$ acts on $\mathbb{R}^4$ by matrix multiplication. Consider the induced action on $\mathcal{S}(\mathbb{R}^4)\otimes\Lambda(\mathbb{R}^4)$, the space of differential forms with coefficients in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. The tensor product $\mathcal{W}\otimes\mathcal{C}$ of the Weyl algebra $\mathcal{W}$, generated by the operators x_j and ∂_{x_i} and the Clifford algebra $\mathcal{C}$ generated by the corresponding operators $\wedge_j$ and $\rfloor_i$ acts on $\mathcal{S}(\mathbb{R}^4)\otimes\Lambda(\mathbb{R}^4)$.

The centralizer of $O_{3,1}$ in $\mathcal{W}\otimes\mathcal{C}$ is spanned by the Minkowski metric $r^2_{3,1}$, the Dalambertian $\Delta_{3,1}$, the Euler operator $\sum_{i=1}^4 x_i\partial_{x_i}$, the exterior degree operator $\sum_{i=1}^4 \rfloor_i\wedge_i$, the exterior differential $d=\sum_{i=1}^4\wedge_i\partial_{x_i}$, it adjoint $\delta=\sum_{i=1}^4 x_i\ \rfloor_i$, and the conjugates of these by the Lorent-Hodge $*$, $*^{-1}d*$, $*^{-1}\delta*$. The point is that they form a Lie superalgebra isomorphic to $\mathfrak{os}_{2,2}=(\mathfrak{sp}_2\oplus\mathfrak{o}_2)\oplus\mathbb{R}^2\otimes\mathbb{R}^2$.

One may restrict the action of the group to the space of forms supported on the light cone in $\mathbb{R}^4$, as in [40], and study resulting vector bundles over the corresponding symmetric space. The above mentioned super Lie algebra may provide differential operators that commute with the group action.

Resonances of the Laplace operator on Whittaker functions. As explained in [31], the generalized spherical functions of a K-type, I used in my thesis to compute the resonances of the Laplace operator have "dual spherical functions" in the "Whittaker world". Instead of considering the quotient by K, we consider its dual space (of horocycle) $\Xi=G/MN$. For this consider a representation σ of MN on a finite dimensional space V_σ. The dual spherical functions are eigenfunctions of each $D\in\mathbf{D}(\Xi)$ and extend by

$$\begin{array}{rccl}\Psi: & MN\times\Xi & \longrightarrow & \mathrm{Hom}(V_\sigma,V_\sigma)\\ & h\xi & \longmapsto & \Psi(h\cdot\xi)=\sigma(h)\Psi(\xi)\ .\end{array}$$

This leads to Whittaker functions and Whittaker distributions. As the Plancherel formula for Whittaker functions has been shown in [67], we can possibly extend the results of [60] to these functions.

Hyperbolic manifolds. The usual step expected in the analysis on G/K is extending it to more general homogenous spaces. One might consider the Laplace operator acting on vector bundles over the orbit space $\Gamma\backslash G/K$, of a discrete torsion-free subgroup $\Gamma\subseteq G$. The basic results on the spectrum of Δ for the bundle of the p-forms on G/K allowed [12] to draw conclusions on the associated locally symmetric situation. I expect that the results of my thesis, which are more general, could also shed additional light on the structure of the resonance spaces.

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